First name:

Last name:

Student number:

- There are 3 questions, worth 20 marks each.
- You have 90 minutes.
- Answer all questions in the spaces provided. You may use the reverse side if more space is required, or more paper is available on request.

Question 1. (20 marks) Let M be the matroid represented over the real numbers by the following matrix:

$$\begin{array}{cccccc} a & b & c & d & e & f \end{array}$$
$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}$$

(a) Give a geometric representation of M. Be sure to clearly label the elements of M.

(b) Find a 3-element circuit, and a 4-element circuit, of M.

(c) Find a 3-element cocircuit, and a 4-element cocircuit, of M.

(d) Find two distinct hyperplanes of M.

(e) Find two distinct bases of M.

(f) Give a geometric representation of M^* and a matrix representation of M^*. For each representation, clearly label the elements of M^*. Comment on the relationship between M and M^*.

(g) Find a coindependent set that is a flat.

(h) Find a cobasis X such that $\mathrm{cl}(X) = X$.

(i) Is M 2-connected? Explain your answer.

(j) Does M have any 3-separations? Explain your answer.

(k) Find a matrix that represents M/a over the reals, and give a geometric representation of M/a. Label the elements in each representation.

(l) Is M graphic? Briefly explain your answer.

Question 2. (20 marks) Definitions and short proofs.

(a) State the circuit axioms for a matroid.

(b) Define what is meant by the *dual* M^* of a matroid M.

(c) Let x be an element of a matroid M. Define the *contraction* M/x of x from M.

(d) State what it means for the rank function of a matroid M to be *submodular*. (In other words, state the third rank axiom.)

(e) Let λ_M be the connectivity function of a matroid M. Show that $\lambda_M(X) \geq 0$ for all $X \subseteq E(M)$.

(f) Let M be a matroid with ground set E. Prove the following:

(i) For $X \subseteq E$, the set X is a coindependent if and only if $E - X$ is spanning.

(ii) For $e \in E$, the element e is a coloop if and only if $E - e$ is a hyperplane.

Question 3. (20 marks)

(a) Consider the uniform matroid $U_{r,n}$. Specify the bases, flats, rank function, and closure operator of $U_{r,n}$.

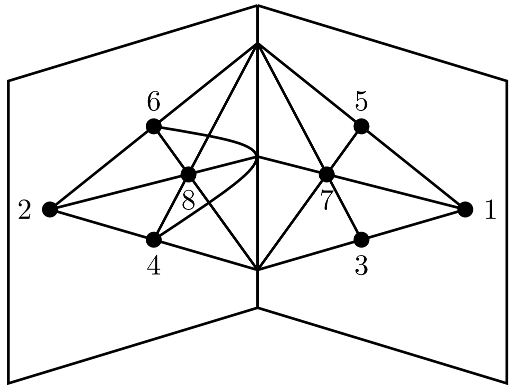
(b) Say $x \in E(U_{r,n})$. Describe $U_{r,n}/x$ and $U_{r,n} \backslash x$.

(c) Construct geometric representations of $U_{1,4}$, $U_{2,5}$, and $U_{3,6}$.

(d) Find an excluded minor for the class of uniform matroids.

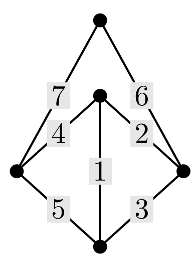
(e) Provide four (pairwise non-isomorphic) matroids that are excluded minors for the class of $GF(5)$-representable matroids.

(f) Consider the rank-4 matroid M with the following geometric representation:



What fields is M representable over, if any? Provide an argument to support your answer.

(g) Consider the following graph G.



Provide a matrix A with all entries in $\{0, 1, -1\}$ such that, for any field $\mathbb{F}$, if A is viewed as a matrix over $\mathbb{F}$, then $M[A] \cong M(G)$. Label the columns of A.