Relation Algebras and $\mathbf{R}$

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Algebras of binary relations

Let $U$ be any set. Consider $\mathcal{P}(U \times U)$ with the following operations:
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- identity relation ($Id$), bottom ($\emptyset$), top ($U \times U$)
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- relational composition ($\circ$) and converse ($^{-1}$)
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The structure $\langle \mathcal{P}(U \times U); \cup, \cap, \circ, \neg, ^{-1}, Id, \emptyset, U \times U \rangle$ is an algebra of binary relations.
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Let \( U \) be any set. Consider \( \mathcal{P}(U \times U) \) with the following operations:

- union (\( \cup \)), intersection (\( \cap \)) and complement (\( ^- \))
- relational composition (\( \circ \)) and converse (\( ^{-1} \))
- identity relation (\( Id \)), bottom (\( \emptyset \)), top (\( U \times U \))

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Algebras of binary relations

Let $U$ be any set. Consider $\wp(U \times U)$ with the following operations:

- union ($\cup$), intersection ($\cap$) and complement ($\neg$)
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The structure $\langle \wp(U \times U); \cup, \cap, \circ, \neg, ^{-1}, Id, \emptyset, U \times U \rangle$ is an algebra of binary relations.

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- E. Schröder, *Vorlesungen über die Algebra der Logik* (1895)
- A. Tarski, *On the calculus of relations* (1941)
Equations satisfied by them

1. Equations making \( \langle \varnothing(U \times U); \cup, \cap, -, Id, \emptyset, U \times U \rangle \) into a Boolean algebra.
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2. Equations making $\langle \wp(U \times U); \circ, -1, Id \rangle$ into an involutive monoid.
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3. Equations making \( \langle \wp(U \times U); \cup, \cap, \circ, -, -^{-1}, \text{Id}, \emptyset, U \times U \rangle \) into a Boolean Algebra with Operators, namely
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   - $x \circ (y \cup z) = x \circ y \cup x \circ z$, $(x \cup y) \circ z = x \circ z \cup y \circ z$
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Equations satisfied by them

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   - \( \emptyset^{-1} = \emptyset \)

4. Triangle laws:

   \[ x \circ y \cap z = \emptyset \text{ iff } x^{-1} \circ z \cap y = \emptyset \text{ iff } z \circ y^{-1} \cap x = \emptyset. \]
Equations satisfied by them

1. Equations making \( \langle \wp(U \times U); \cup, \cap, -, Id, \emptyset, U \times U \rangle \) into a Boolean algebra.

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4. Triangle laws:
   \[
x \circ y \cap z = \emptyset \text{ iff } x^{-1} \circ z \cap y = \emptyset \text{ iff } z \circ y^{-1} \cap x = \emptyset.
   \]

5. Or, equivalently, \((x^{-1} \circ (x \circ y)^{-1}) \cup y^{-1} = y^{-}\).
Abstract relation algebras

Definition (Jónsson and Tarski, 1948)

An abstract relation algebra (RA) is any algebra $A = \langle A; \lor, \land, \cdot, -, -^{-1}, e, 0, 1 \rangle$ satisfying the equations from the previous slide.
Abstract relation algebras

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An abstract RA $A$ is a representable RA (RRA) if $A$ can be embedded in a direct product of concrete algebras of binary relations.
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**Theorem (Lyndon, 1950)**

*There are non-representable relation algebras.*
Abstract relation algebras

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Theorem (Lyndon, 1950)

There are non-representable relation algebras.

So, Lyndon’s result reads: \( RRA \subset RA \). But still,

Theorem (Tarski, 1955)

The class RRA is a variety.
Positive $\mathbf{R}$

Axioms:

1. $\alpha \rightarrow \alpha$
2. $(\alpha \rightarrow \beta) \rightarrow ((\gamma \rightarrow \alpha) \rightarrow (\gamma \rightarrow \beta))$
3. $((\alpha \rightarrow (\alpha \rightarrow \beta)) \rightarrow (\alpha \rightarrow \beta)$
4. $(\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow (\beta \rightarrow (\alpha \rightarrow \gamma))$

Rules:

\[
\begin{array}{c}
\alpha \\
\alpha \rightarrow \beta \\
\hline
\beta 
\end{array}
\]
Positive $\mathbf{R}$

Axioms:

1. $\alpha \to \alpha$
2. $(\alpha \to \beta) \to ((\gamma \to \alpha) \to (\gamma \to \beta))$
3. $((\alpha \to (\alpha \to \beta)) \to (\alpha \to \beta))$
4. $(\alpha \to (\beta \to \gamma)) \to (\beta \to (\alpha \to \gamma))$
5. $(\alpha \wedge \beta) \to \alpha$, $(\alpha \wedge \beta) \to \beta$
6. $((\alpha \to \beta) \wedge (\alpha \to \gamma)) \to (\alpha \to (\beta \wedge \gamma))$

Rules:

$$
\frac{\alpha \quad \alpha \to \beta}{\beta} \quad \frac{\alpha \quad \beta}{\alpha \wedge \beta}
$$
Positive \( R \)

Axioms:

1. \( \alpha \rightarrow \alpha \)
2. \( (\alpha \rightarrow \beta) \rightarrow ((\gamma \rightarrow \alpha) \rightarrow (\gamma \rightarrow \beta)) \)
3. \( ((\alpha \rightarrow (\alpha \rightarrow \beta)) \rightarrow (\alpha \rightarrow \beta) \)
4. \( (\alpha \rightarrow (\beta \rightarrow \gamma)) \rightarrow (\beta \rightarrow (\alpha \rightarrow \gamma)) \)
5. \( (\alpha \land \beta) \rightarrow \alpha, (\alpha \land \beta) \rightarrow \beta \)
6. \( ((\alpha \rightarrow \beta) \land (\alpha \rightarrow \gamma)) \rightarrow (\alpha \rightarrow (\beta \land \gamma)) \)
7. \( \alpha \rightarrow (\alpha \lor \beta), \beta \rightarrow (\alpha \lor \beta) \)
8. \( ((\alpha \rightarrow \gamma) \land (\beta \rightarrow \gamma)) \rightarrow ((\alpha \lor \beta) \rightarrow \gamma) \)

Rules:

\[
\begin{array}{ccc}
\alpha & \alpha \rightarrow \beta & \alpha \\
\hline
\beta & \beta & \alpha \land \beta \\
\end{array}
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Positive \( \mathbf{R} \)

Axioms:

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5. \( (\alpha \land \beta) \rightarrow \alpha, (\alpha \land \beta) \rightarrow \beta \)
6. \( ((\alpha \rightarrow \beta) \land (\alpha \rightarrow \gamma)) \rightarrow (\alpha \rightarrow (\beta \land \gamma)) \)
7. \( \alpha \rightarrow (\alpha \lor \beta), \beta \rightarrow (\alpha \lor \beta) \)
8. \( ((\alpha \rightarrow \gamma) \land (\beta \rightarrow \gamma)) \rightarrow ((\alpha \lor \beta) \rightarrow \gamma) \)
9. \( (\alpha \land (\beta \lor \gamma)) \rightarrow ((\alpha \land \beta) \lor \gamma) \)

Rules:

\[
\frac{\alpha \quad \alpha \rightarrow \beta}{\beta} \quad \frac{\alpha \quad \beta}{\alpha \land \beta}
\]
**R and \( R_t \)**

**Adding negation:**

- \((\alpha \rightarrow \lnot \alpha) \rightarrow \lnot \alpha\)
- \((\alpha \rightarrow \lnot \beta) \rightarrow (\beta \rightarrow \lnot \alpha)\)
- \(\lnot \lnot \alpha \rightarrow \alpha\)
**R and R**

Adding negation:

- \((\alpha \rightarrow \neg \alpha) \rightarrow \neg \alpha\)
- \((\alpha \rightarrow \neg \beta) \rightarrow (\beta \rightarrow \neg \alpha)\)
- \(\neg \neg \alpha \rightarrow \alpha\)

Adding fusion (for **R**; for weaker logics a rule is needed instead)

- \(((\alpha \cdot \beta) \rightarrow \gamma) \leftrightarrow (\alpha \rightarrow (\beta \rightarrow \gamma))\)
Adding negation:

> $\alpha \to \neg \alpha \to \neg \alpha$
> $\alpha \to \neg \beta \to (\beta \to \neg \alpha)$
> $\neg \neg \alpha \to \alpha$

Adding fusion (for $R$; for weaker logics a rule is needed instead)

> $((\alpha \cdot \beta) \to \gamma) \leftrightarrow (\alpha \to (\beta \to \gamma))$

——— this separates $R$ (above) from $R_t$ (below) ————
R and R_t

Adding negation:

\[
\begin{align*}
\triangleright (\alpha \rightarrow \neg\alpha) & \rightarrow \neg\alpha \\
\triangleright (\alpha \rightarrow \neg\beta) & \rightarrow (\beta \rightarrow \neg\alpha) \\
\triangleright \neg\neg\alpha & \rightarrow \alpha 
\end{align*}
\]

Adding fusion (for R; for weaker logics a rule is needed instead)

\[
\begin{align*}
\triangleright ((\alpha \cdot \beta) \rightarrow \gamma) & \leftrightarrow (\alpha \rightarrow (\beta \rightarrow \gamma)) \\
\end{align*}
\]

this separates R (above) from R_t (below)

Adding Ackermann constant t:

\[
\begin{align*}
\triangleright t \\
\triangleright t \rightarrow (\alpha \rightarrow \alpha)
\end{align*}
\]
R and $R_t$

Adding negation:

$\alpha \rightarrow \neg \alpha \rightarrow \neg \alpha$

$\alpha \rightarrow \neg \beta \rightarrow (\beta \rightarrow \neg \alpha)$

$\neg \neg \alpha \rightarrow \alpha$

Adding fusion (for $R$; for weaker logics a rule is needed instead)

$((\alpha \cdot \beta) \rightarrow \gamma) \leftrightarrow (\alpha \rightarrow (\beta \rightarrow \gamma))$

This separates $R$ (above) from $R_t$ (below)

Adding Ackermann constant $t$:

$t$

$t \rightarrow (\alpha \rightarrow \alpha)$

All these are conservative extensions.
De Morgan monoids defined

A **De Morgan monoid** is an algebra $\mathbf{M} = \langle M, \lor, \land, \cdot, \to, 1, 0 \rangle$ of type $\langle 2, 2, 2, 2, 0, 0 \rangle$ such that:

1. $(M, \lor, \land)$ is a distributive lattice,
2. $(M, \cdot, 1)$ is a commutative monoid,
3. the residuation equivalence: $a \cdot b \leq c$ iff $b \leq a \to c$, holds for all $a, b, c \in M$,
4. the following identities hold in $M$: $a \leq a \cdot a \iff (a \to 0) \to 0 = a$.

Negation is defined in $M$ by $\neg a = a \to 0$. 

**Lemma**

De Morgan monoids form a finitely based variety.
De Morgan monoids defined

A De Morgan monoid is an algebra $\mathbf{M} = \langle M, \lor, \land, \cdot, \rightarrow, 1, 0 \rangle$ of type $\langle 2, 2, 2, 2, 0, 0 \rangle$ such that:

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Negation is defined in $\mathbf{M}$ by $\sim a = a \to 0$. 
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4. the following identities hold in \( M \):
   
   - \( a \leq a \cdot a \)
   - \( (a \to 0) \to 0 = a \)

Negation is defined in \( M \) by \( \sim a = a \to 0 \).

**Lemma**

*De Morgan monoids form a finitely based variety.*
De Morgan monoids redefined

- A bare De Morgan monoid is \(\{1, 0\}\)-free reduct of an algebra \(\langle M, \lor, \land, \cdot, \to, \sim, 0, 1 \rangle\), where \(\langle M, \lor, \land, \cdot, \to, 0, 1 \rangle\) is a De Morgan monoid and \(\sim\) is the negation on \(M\).
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- A (bare) De Morgan monoid \(M\) is normal if for every \(a \in M\) exactly one of \(a, \sim a\) is greater or equal to 1.
De Morgan monoids redefined

- A bare De Morgan monoid is \( \{1, 0\}\)-free reduct of an algebra \( \langle M, \vee, \wedge, \cdot, \rightarrow, \sim, 0, 1 \rangle \), where \( \langle M, \vee, \wedge, \cdot, \rightarrow, 0, 1 \rangle \) is a De Morgan monoid and \( \sim \) is the negation on \( M \).

- A (bare) De Morgan monoid \( M \) is normal if for every \( a \in M \) exactly one of \( a, \sim a \) is greater or equal to 1.

**Theorem**

- \( R_t \) is sound and complete with respect to normal De Morgan monoids.

- \( R \) is sound and complete with respect to normal bare De Morgan monoids.
Soundness

In 2007, Roger Maddux observed that defining

\[ x \rightarrow y = (y^{-} \cdot x^{-1})^{-} \]
\[ \sim x = (x^{-1})^{-} \]

makes all axioms and rules of \( \mathbf{R} \) true in any square increasing commutative representable relation algebra (\( \equiv \) algebra of dense commuting relations).
Soundness

In 2007, Roger Maddux observed that defining

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\[
\begin{align*}
\star & x \rightarrow y = (y^{-} \cdot x^{-1})^{-} \\
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Question (Maddux)

*Is \( R \) also complete with respect to square-increasing, commutative, representable relation algebras?*
Some incompleteness, some completeness

On the negative side, Mikulás Szabolc proved:

**Theorem (Szabolc, 2008)**

*There are formulae that hold in square-increasing, commutative, representable relation algebras, but fail to be theorems of $\mathbf{R}$. Thus, $\mathbf{R}$ is not complete with respect to square-increasing, commutative RRAs.*
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There areformulae that hold in square-increasing, commutative, representable relation algebras, but fail to be theorems of $\mathbf{R}$. Thus, $\mathbf{R}$ is not complete with respect to square-increasing, commutative RRAs.

On the positive side, Roger Maddux showed:

**Theorem (Maddux, 2010)**

$\mathbf{RM}$ is sound and complete with respect to square-increasing, transitive, representable relation algebras.
Maddux’s question asked again

**Question**

*Is $\mathbf{R}$ complete with respect to square-increasing, commutative (but not necessarily representable) relation algebras?*
Maddux’s question asked again

Question

Is $\mathbf{R}$ complete with respect to square-increasing, commutative (but not necessarily representable) relation algebras?

To answer it in the positive, it would suffice to find an embedding

$$\varepsilon : \mathbf{M} \rightarrow \mathbf{A}$$

sending an arbitrary normal bare De Morgan monoid $\mathbf{M}$ into a square-increasing, commutative relation algebra $\mathbf{A}$. 
Maddux’s question asked again

**Question**

*Is R complete with respect to square-increasing, commutative (but not necessarily representable) relation algebras?*

To answer it in the positive, it would suffice to find an embedding

\[ \varepsilon : M \rightarrow A \]

sending an arbitrary normal bare De Morgan monoid M into a square-increasing, commutative relation algebra A. Which is what we are going to do next.
Subdirectly irreducible De Morgan monoids

On a De Morgan monoid $\mathbf{M}$, we define the dual of multiplication, putting $a + b = \sim(\sim a \cdot \sim b)$ for all $a, b \in M$. 
Subdirectly irreducible De Morgan monoids

On a De Morgan monoid $M$, we define the dual of multiplication, putting $a + b = \sim(\sim a \cdot \sim b)$ for all $a, b \in M$.

Lemma

Let $M$ be a subdirectly irreducible De Morgan monoid, and $a, b, c, d$ be elements of $M$ such that $ab \leq c + d$. Then $a \leq c$ or $b \leq d$. 
Subdirectly irreducible De Morgan monoids

On a De Morgan monoid $M$, we define the dual of multiplication, putting $a + b = \sim(\sim a \cdot \sim b)$ for all $a, b \in M$.

**Lemma**

Let $M$ be a subdirectly irreducible De Morgan monoid, and $a, b, c, d$ be elements of $M$ such that $ab \leq c + d$. Then $a \leq c$ or $b \leq d$.

Compare this with the Boolean case, where $x \cdot y = x \land y$ and $x + y = x \lor y$.

**Lemma**

Let $2$ be the two-element Boolean algebra, and $a, b, c, d \in \{0, 1\}$ be such that $a \land b \leq c \lor d$. Then $a \leq c$ or $b \leq d$. 
Prime filters

A proper filter $F$ on a lattice $L$ is prime if $a \lor b \in F$ implies $a \in F$ or $b \in F$, for any $a, b \in L$.
Dually, a proper ideal $I$ on $L$ is prime if $a \land b \in I$ implies $a \in I$ or $b \in I$, for any $a, b \in L$.

**Lemma**

Let $L$ be a distributive lattice, $F \subseteq L$ and $a \in L$. Then, the following hold.

- If $F$ is a prime filter, then $\overline{F}$ is a prime ideal.
- If $F$ is a filter and $a \notin F$, then there is a prime filter $P$ such that $F \subseteq P$ and $a \notin P$.

Prime filters on De Morgan monoids are even better.
Prime filters on De Morgan monoids

Lemma

Let $\mathbf{M}$ be a De Morgan monoid and $a, b \in \mathbf{M}$. Let $F, G, H, P$ be prime filters on $\mathbf{M}$. The following hold:

1. If $FGH \subseteq P$ and $\overline{F} + \overline{G} + \overline{H} \subseteq \overline{P}$, then there exists a prime filter $R$ such that $GH \subseteq R$, $\overline{G} + \overline{H} \subseteq \overline{R}$, $FR \subseteq P$, and $\overline{F} + \overline{R} \subseteq \overline{P}$.

2. If $FGH \subseteq P$, then there exists a prime filter $Q$ such that $FG \subseteq Q$ and $QH \subseteq P$.

3. If $ab \in P$, then there exist prime filters $Q$ and $R$ such that $a \in Q$, $b \in R$, $QR \subseteq P$, and $\overline{Q} + \overline{R} \subseteq \overline{P}$. 


Prime filters, prime ideals and negation

Lemma

Let $\mathbf{M}$ be a De Morgan monoid, $F$ a prime filter on $\mathbf{M}$, and $I$ a prime ideal on $\mathbf{M}$. Then, the following hold:

1. $\sim F = \{ \sim a \in M : a \in F \}$ is a prime ideal.
2. $\overline{F} = \{ a \in F : a \not\in F \}$ is a prime ideal.
3. $\sim I = \{ \sim a \in M : a \in I \}$ is a prime filter.
4. $\overline{I} = \{ a \in F : a \notin I \}$ is a prime filter.
5. $\overline{\sim F} = \sim \overline{F}$ is a prime filter.
6. $\overline{\sim I} = \sim \overline{I}$ is a prime ideal.
7. If $\mathbf{M}$ is normal, then $\uparrow 1 = \{ a \in M : a \geq 1 \}$ is a prime filter.
RAs from De Morgan monoids: first step

Let $\mathcal{F}$ be the set of all prime filters of a subdirectly irreducible De Morgan monoid $\mathbf{M}$. $U = \wp(\mathcal{F})$ is a Boolean algebra under the set-theoretical operations $\cup$, $\cap$ and $\neg$. The atoms of $U$ are of the form $\{F\}$, for $F \in \mathcal{F}$. Define a partial multiplication $\circ$ on $U$, by

$$\{F\} \circ \{G\} = \{P \in \mathcal{F} : FG \subseteq P, \overline{F} + \overline{G} \subseteq \overline{P}\}.$$
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**Lemma**

Let $F, G \in \mathcal{F}$. Then $\{F\} \circ \{G\} \neq \emptyset$. 
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*Let $F, G \in \mathcal{F}$. Then $\{F\} \circ \{G\} \neq \emptyset$.*

It makes the multiplication “integral”, that is, having no “zero-divisors”. This works for any si De Morgan monoid.
RAs from De Morgan monoids: multiplication

If $M$ is normal, we also have

$$\{\uparrow 1\} \circ \{F\} = \{F\} = \{F\} \circ \{\uparrow 1\}$$

so $\{\uparrow 1\}$ acts as an identity for $\circ$. 
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$$u \circ w = \bigcup \{a \circ b : a, b \in \text{At}(U), a \leq u, b \leq w\}$$

we extend multiplication onto the whole $U$. 
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**Lemma**

*Multiplication on $U$ defined as above is associative (and commutative). Thus, $\langle U; \circ, \{\uparrow 1\} \rangle$ is a (commutative) monoid.*
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Lemma

*Multiplication on $U$ defined as above is associative (and commutative). Thus, $\langle U; \circ, \{\uparrow 1\} \rangle$ is a (commutative) monoid.*

Notice that $\{\uparrow 1\}$ is an atom of $U$. 
RAs from De Morgan monoids: converse

Define a unary operation $^{-1}$ on $U$ putting

$$\{F\}^{-1} = \{\sim F\}$$

for the atoms, and then set

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$\langle U, \cup, \cap, \ominus, ^{-1}, -, e, 0, 1 \rangle$, where $\cup, \cap, -$ are the usual operations on $U = \wp(F)$, and $e = \{1\}, 0 = \emptyset, 1 = F$. 
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**Theorem**

*The algebra $\mathbf{U}$ is a square-increasing relation algebra.*
The embedding

Let $M$ be a normal De Morgan monoid, and $U_M$ the relation algebra defined in the previous slide. Consider a map $\varepsilon: M \to U_M$ defined by

$$\varepsilon(a) = \{ F \in \mathcal{F} : a \in F \}$$
The embedding

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Lemma

For any $a \in M$, the following hold:

1. $\varepsilon(a \lor b) = \varepsilon(a) \lor \varepsilon(b)$
2. $\varepsilon(a \land b) = \varepsilon(a) \land \varepsilon(b)$
3. $\varepsilon(ab) = \varepsilon(a) \circ \varepsilon(b)$
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Let $\mathbf{M}$ be a normal De Morgan monoid, and $\mathbf{U}_\mathbf{M}$ the relation algebra defined in the previous slide. Consider a map $\varepsilon : \mathbf{M} \rightarrow \mathbf{U}_\mathbf{M}$ defined by

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3. $\varepsilon(ab) = \varepsilon(a) \circ \varepsilon(b)$
4. $\varepsilon(\sim a) = (\varepsilon(a)^{-1})^-$
5. $\varepsilon(a \rightarrow b) = (\varepsilon(b)^- \circ \varepsilon(a)^{-1})^-$
Completeness

**Theorem**

*Every normal De Morgan monoid is embeddable as a bare De Morgan monoid into a square-increasing, commutative, integral, simple relation algebra.*
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The embedding cannot extend to full De Morgan monoids.
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_Every normal De Morgan monoid is embeddable as a bare De Morgan monoid into a square-increasing, commutative, integral, simple relation algebra._

The embedding cannot extend to full De Morgan monoids. For

- \( \varepsilon(1) = \{ F \in \mathcal{F} : 1 \in F \} \neq e = \{ \uparrow 1 \} \), and
- \( \{ F \in \mathcal{F} : 1 \in F \} \neq \{ F \in \mathcal{F} : 1 \in F \}^{-1} = \{ F \in \mathcal{F} : 0 \notin F \} \)

so, \( \varepsilon(1) \) is not even a symmetric element.
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Theorem

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Corollary

R is sound and complete with respect to square-increasing, commutative relation algebras.
Normal varieties of De Morgan monoids

**Theorem**

Let $\mathbf{M}$ be a De Morgan monoid. The following are equivalent:

1. $\mathbf{M} \models 1 \leq 0 \lor x \Rightarrow 1 \leq x$
2. $\mathbf{M}$ has a subdirect representation with normal factors.
Normal varieties of De Morgan monoids

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We will say that a variety $\mathcal{V}$ of De Morgan monoids is normal if it is generated by normal De Morgan monoids.
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**Theorem**

$V$ is normal iff $F_V(x)\models 1 \leq 0 \lor x \Rightarrow 1 \leq x$. 
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Let $M$ be a De Morgan monoid. The following are equivalent:

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**Theorem**

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**Theorem**

Normal varieties of De Morgan monoids form a complete sublattice of the lattice of all varieties of De Morgan monoids.
Embedding for normal varieties

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- $\mathcal{V}$ — a normal variety of De Morgan monoids,
Embedding for normal varieties

Consider:

- \( \mathcal{V} \) — a normal variety of De Morgan monoids,
- \( \mathcal{V}_N \) — the class of normal members of \( \mathcal{V} \),
Embedding for normal varieties

Consider:

- $\mathcal{V}$ — a normal variety of De Morgan monoids,
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Reasonable conjecture

The map $\rho$ is a lattice embedding.
Embedding for normal varieties

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Crazy conjecture

$\rho(\mathcal{V})$ is axiomatised by “Maddux translations” of the identities axiomatising $\mathcal{V}$. 