Relation Algebras and **R**

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5. Or, equivalently, $(x^{-1} \circ (x \circ y)^-) \cup y^- = y^-$.

Abstract relation algebras

Definition (Jónsson and Tarski, 1948)

An abstract relation algebra (RA) is any algebra $\mathbf{A} = \langle A; \vee, \wedge, \cdot, ^{-}, ^{-1}, e, 0, 1\rangle$ satisfying the equations from the previous slide.

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So, Lyndon's result reads: $RRA \subset RA$. But still,

Theorem (Tarski, 1955)

The class RRA is a variety.

Positive **R**

Axioms:

1. $\alpha \to \alpha$
2. $(\alpha \to \beta) \to \big((\gamma \to \alpha) \to (\gamma \to \beta)\big)$
3. $\big((\alpha \to (\alpha \to \beta) \to (\alpha \to \beta)$
4. $\big(\alpha \to (\beta \to \gamma)\big) \to \big(\beta \to (\alpha \to \gamma)\big)$

Rules:

$$\frac{\alpha \quad \alpha \to \beta}{\beta}$$

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R and $\mathbf{R}_t$

Adding negation:

- $(\alpha \to \neg\alpha) \to \neg\alpha$
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Adding fusion (for **R**; for weaker logics a rule is needed instead)

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Adding Ackermann constant t:

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- $t \to (\alpha \to \alpha)$

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All these are conservative extensions.

De Morgan monoids defined

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Lemma

De Morgan monoids form a finitely based variety.

De Morgan monoids redefined

- A bare De Morgan monoid is $\{1, 0\}$-free reduct of an algebra $\langle M, \vee, \wedge, \cdot, \rightarrow, \sim, 0, 1\rangle$, where $\langle M, \vee, \wedge, \cdot, \rightarrow, 0, 1\rangle$ is a De Morgan monoid and $\sim$ is the negation on $\mathbf{M}$.

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- A (bare) De Morgan monoid $\mathbf{M}$ is normal if for every $a \in M$ exactly one of $a, \sim a$ is greater or equal to 1.

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Theorem

- $\mathbf{R}_t$ *is sound and complete with respect to normal De Morgan monoids.*
- $\mathbf{R}$ *is sound and complete with respect to normal bare De Morgan monoids.*

Soundness

In 2007, Roger Maddux observed that defining

- $x \to y = (y^- \cdot x^{-1})^-$
- $\sim x = (x^{-1})^-$

makes all axioms and rules of **R** true in any square increasing commutative representable relation algebra (= algebra of dense commuting relations).

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Question (Maddux)

Is **R** *also complete with respect to square-increasing, commutative, representable relation algebras?*

Some incompleteness, some completeness

On the negative side, Mikulás Szabolc proved:

Theorem (Szabolc, 2008)

There are formulae that hold in square-increasing, commutative, representable relation algebras, but fail to be theorems of **R**. *Thus,* **R** *is not complete with respect to square-increasing, commutative RRAs.*

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On the positive side, Roger Maddux showed:

Theorem (Maddux, 2010)

RM *is sound and complete with respect to square-increasing, transitive, representable relation algebras.*

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Is **R** *complete with respect to square-increasing, commutative (but not necessarily representable) relation algebras?*

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To answer it in the positive, it would suffice to find an embedding

$$\varepsilon\colon \mathbf{M} \to \mathbf{A}$$

sending an arbitrary normal bare De Morgan monoid **M** into a square-increasing, commutative relation algebra **A**.

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To answer it in the positive, it would suffice to find an embedding

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sending an arbitrary normal bare De Morgan monoid **M** into a square-increasing, commutative relation algebra **A**. Which is what we are going to do next.

Subdirectly irreducible De Morgan monoids

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Lemma

Let **M** *be a subdirectly irreducible De Morgan monoid, and* a, b, c, d *be elements of* M *such that* $ab \leq c + d$*. Then* $a \leq c$ *or* $b \leq d$*.*

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Compare this with the Boolean case, where $x \cdot y = x \wedge y$ and $x + y = x \vee y$.

Lemma

Let **2** *be the two-element Boolean algebra, and* $a, b, c, d \in \{0, 1\}$ *be such that* $a \wedge b \leq c \vee d$*. Then* $a \leq c$ *or* $b \leq d$*.*

Prime filters

A proper filter F on a lattice $\mathbf{L}$ is prime if $a \vee b \in F$ implies $a \in F$ or $b \in F$, for any $a, b \in L$.
Dually, a proper ideal I on $\mathbf{L}$ is prime if $a \wedge b \in I$ implies $a \in I$ or $b \in I$, for any $a, b \in L$.

Lemma

Let $\mathbf{L}$ be a distributive lattice, $F \subseteq L$ and $a \in L$. Then, the following hold.

- *If F is a prime filter, then $\overline{F}$ is a prime ideal.*
- *If F is a filter and $a \notin F$, then there is a prime filter P such that $F \subseteq P$ and $a \notin P$.*

Prime filters on De Morgan monoids are even better.

Prime filters on De Morgan monoids

Lemma

Let $\mathbf{M}$ *be a De Morgan monoid and* $a, b \in M$*. Let* F, G, H, P *be prime filters on* $\mathbf{M}$*. The following hold:*

1. *If* $FGH \subseteq P$ *and* $\overline{F} + \overline{G} + \overline{H} \subseteq \overline{P}$*, then there exists a prime filter* R *such that* $GH \subseteq R$*,* $\overline{G} + \overline{H} \subseteq \overline{R}$*,* $FR \subseteq P$*, and* $\overline{F} + \overline{R} \subseteq \overline{P}$*.*
2. *If* $FGH \subseteq P$*, then there exists a prime filter* Q *such that* $FG \subseteq Q$ *and* $QH \subseteq P$*.*
3. *If* $ab \in P$*, then there exist prime filters* Q *and* R *such that* $a \in Q$*,* $b \in R$*,* $QR \subseteq P$*, and* $\overline{Q} + \overline{R} \subseteq \overline{P}$*.*

Prime filters, prime ideals and negation

Lemma

Let $\mathbf{M}$ *be a De Morgan monoid,* F *a prime filter on* $\mathbf{M}$*, and* I *a prime ideal on* $\mathbf{M}$*. Then, the following hold:*

1. $\sim F = \{\sim a \in M \colon a \in F\}$ *is a prime ideal.*
2. $\overline{F} = \{a \in F \colon a \notin F\}$ *is a prime ideal.*
3. $\sim I = \{\sim a \in M \colon a \in I\}$ *is a prime filter.*
4. $\overline{I} = \{a \in F \colon a \notin I\}$ *is a prime filter.*
5. $\overline{\sim F} = \sim \overline{F}$ *is a prime filter.*
6. $\overline{\sim I} = \sim \overline{I}$ *is a prime ideal.*
7. *If* $\mathbf{M}$ *is normal, then* $\uparrow 1 = \{a \in M \colon a \geq 1\}$ *is a prime filter.*

RAs from De Morgan monoids: first step

Let $\mathcal{F}$ be the set of all prime filters of a subdirectly irreducible De Morgan monoid $\mathbf{M}$. $U = \wp(\mathcal{F})$ is a Boolean algebra under the set-theoretical operations $\cup$, $\cap$ and $-$. The atoms of U are of the form $\{F\}$, for $F \in \mathcal{F}$. Define a partial multiplication $\circ$ on U, by

$$\{F\} \circ \{G\} = \{P \in \mathcal{F} \colon FG \subseteq P,\ \overline{F} + \overline{G} \subseteq \overline{P}\}.$$

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It makes the multiplication "integral", that is, having no "zero-divisors". This works for any si De Morgan monoid.

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If **M** is normal, we also have

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Notice that $\{\uparrow 1\}$ is an atom of U.

RAs from De Morgan monoids: converse

Define a unary operation $^{-1}$ on U putting

$$\{F\}^{-1} = \{\overline{\sim F}\}$$

for the atoms, and then set

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Theorem

The algebra $\mathbf{U}$ *is a square-increasing relation algebra.*

The embedding

Let $\mathbf{M}$ be a normal De Morgan monoid, and $\mathbf{U_M}$ the relation algebra defined in the previous slide. Consider a map $\varepsilon\colon M \to U_{\mathbf{M}}$ defined by

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For any $a \in M$, the following hold:

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4. $\varepsilon(\sim a) = (\varepsilon(a)^{-1})^{-}$
5. $\varepsilon(a \to b) = \left(\varepsilon(b)^{-} \circ \varepsilon(a)^{-1}\right)^{-}$

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Theorem

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Corollary

R *is sound and complete with respect to square-increasing, commutative relation algebras.*

Normal varieties of De Morgan monoids

Theorem

Let **M** *be a De Morgan monoid. The following are equivalent:*

1. $\mathbf{M} \models 1 \leq 0 \vee x \;\Rightarrow\; 1 \leq x$
2. **M** *has a subdirect representation with normal factors.*

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Normal varieties of De Morgan monoids form a complete sublattice of the lattice of all varieties of De Morgan monoids.

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Crazy conjecture

$\rho(\mathcal{V})$ is axiomatised by "Maddux translations" of the identities axiomatising $\mathcal{V}$.