Victoria University of Wellington

Te Whare Wānanga o te Ūpoko o te Ika a Maui

Lecture 1 of 4: Zero-point energy

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AotearoaFP16: Quantum Field Theory

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Outline:

- Any quantum simple harmonic oscillator has zero-point energy:

$$E_0 = \frac{1}{2}\hbar\omega.$$

- There are many physical situations in which this zero-point energy is undoubtedly real and physical.
- There are other physical situations in which this zero-point energy has a much more ambiguous status.
- Lead in to my subsequent lectures:
  - Zero-point energies in QFT.
  - Casimir energies.
  - Sakharov-style induced gravity.
- Key message:
  — Whenever possible concentrate on physical energy differences.

Outline:

Introduction

Introduction:

Theorem

For the quantum SHO:

$$E_n = \left(n + \frac{1}{2}\right) \hbar\omega$$

Theorem

For the quantum SHO the lowest energy state is:

$$E_0 = \frac{1}{2}\,\hbar\omega$$

This is often called the zero-point energy.

- What is the zero-point energy good for?

Bound states

Bound states

- Consider a non-positive potential that asymptotes to zero:

$$V(x) \leq 0; \qquad V(\pm\infty) \to 0.$$

- Approximate the potential around its local minima by:

$$V(x) = -\Delta + \frac{1}{2} V_0'' \, (x - x_0)^2 + \dots$$

- Approximate bound state energies are:

$$E_n \approx -\Delta + \left(n + \frac{1}{2}\right) \hbar \sqrt{\frac{V_0''}{m}}; \qquad E_n < 0.$$

- The number of bound states is approximately:

$$N \approx \frac{\Delta}{\hbar\omega} + \frac{1}{2} = \frac{\Delta}{\hbar} \sqrt{\frac{m}{V_0''}} + \frac{1}{2}.$$

- To get at least one bound state $(n = 0,\ N = 1)$ we need:

$$\Delta \gtrsim \frac{1}{2}\hbar\omega = \frac{1}{2}\hbar \sqrt{\frac{V_0''}{m}}.$$

- To prevent bound states:

$$\Delta \lesssim \frac{1}{2}\hbar\omega = \frac{1}{2}\hbar\sqrt{\frac{V_0''}{m}}.$$

- Zero-point energy controls the number of bound states... (at least approximately)...

- Example — sech^2 potential (simplified Pöschl–Teller potential):

$$V(x) = -\frac{\Delta}{\cosh^2(x/a)} \approx -\Delta + \frac{\Delta}{a^2}\, x^2 + \dots$$

- Estimate number of bound states using ZPE:

$$N \approx \frac{\Delta}{\hbar\omega} + \frac{1}{2} = \frac{\Delta}{\hbar}\sqrt{\frac{m}{V_0''}} + \frac{1}{2} = \frac{1}{2}\left(\frac{\sqrt{2\Delta m a^2}}{\hbar} + 1\right).$$

- Define:

$$\Delta = \frac{\hbar^2}{2ma^2}\lambda(\lambda - 1)$$

- Exact eigen-energies (standard exercise):

$$E_n = -\frac{\hbar^2}{2ma^2}(\lambda - 1 - n)^2; \qquad 0 \leq n \leq \lambda - 1$$

- Exact number of bound states:

$$N = \text{floor}(\lambda) = \text{floor}\left[\frac{1}{2}\left(\frac{\sqrt{2\Delta ma^2}}{\hbar} + 1\right)\right]$$

- Number of bound states, (if not precise eigen-energies), agrees with the estimate from zero-point energy...

Bound states

- There are many excruciatingly precise and technical theorems on the existence/non-existence/number of bound states...
- But the zero-point energy argument gets to the heart of the matter quickly and incisively...
- Zero-point energies are physically real...

Lamb shift

- To see that there is a non-zero Lamb shift is deceptively easy...
- Energy difference:

$$\Delta E = E\left({}^2S_{1/2}\right) - E\left({}^2P_{1/2}\right).$$

- To estimate the numerical value is much more subtle...
- Key trick:
  Assume electrons are "buffeted" by zero-point fluctuations and write:

$$V = \frac{e^2}{r} \to \frac{e^2}{|\vec{r} + \vec{r}_{zpf}|} = \exp\left(\vec{r}_{zpf} \cdot \nabla\right) \frac{e^2}{r}$$

$$\approx \left\{ 1 + (\vec{r}_{zpf} \cdot \nabla) + \frac{1}{2} (\vec{r}_{zpf} \cdot \nabla)^2 + \dots \right\} \frac{e^2}{r}$$

- Aside:
  Taylor series:

$$\exp(\vec{a} \cdot \vec{\nabla}) \, f(\vec{x}) = f(\vec{x} + \vec{a})$$

- You may not have seen it presented this way before...
- But this really is Taylor's theorem for C^ω (analytic) functions...
- Simply expand the exponential on the LHS...

- Average over zero point fluctuations, making minimal (isotropy) assumptions:

$$\langle \vec{r}_{zpf} \rangle = 0; \qquad \langle \vec{r}_{zpf} \otimes \vec{r}_{zpf} \rangle = \frac{1}{3} \langle (\vec{r}_{zpf})^2 \rangle \; I_{3\times 3}.$$

- Then:

$$\langle V \rangle \approx \left\{ 1 + \frac{1}{6} \langle (\vec{r}_{zpf})^2 \rangle \nabla^2 + \dots \right\} \frac{e^2}{r}$$

$$\approx \frac{e^2}{r} - \frac{1}{6} \langle (\vec{r}_{zpf})^2 \rangle \; 4\pi \; \delta^3(\vec{r}) + \dots$$

- This implies *s*-wave slightly less tightly bound than *p*-wave...

$$\Delta E = \frac{2\pi}{3} \langle (\vec{r}_{zpf})^2 \rangle \; |\psi(0)|^2 ...$$

- So "all" you now need to do is to estimate $\langle (\vec{r}_{zpf})^2 \rangle$....
- Have fun...
- With a little work:

$$\langle (\vec{r}_{zpf})^2 \rangle \approx \frac{1}{2\pi^2} \alpha \left(\frac{\hbar}{mc} \right)^2 \ln(4/\alpha).$$

- With a little more work

$$\Delta E = E\left({}^2S_{1/2}\right) - E\left({}^2P_{1/2}\right) \approx \frac{\alpha^5 mc^2}{6\pi} \ln\left(\frac{1}{\pi\alpha}\right).$$

- Clear message:
  Zero-point fluctuations have real physical effects...

Casimir effect

- Introduce some conducting boundaries...
- Solve for the eigen-frequencies ω_n.
- Formally sum:

$$E_{Casimir} = \sum_n \frac{1}{2}\hbar\omega_n$$

- Being more careful, compare two suitably chosen systems ...
- Calculate Casimir energy differences:

$$\Delta E_{Casimir} = \sum_n \frac{1}{2}\hbar\left(\omega_n - \omega_n^*\right)$$

- See Lecture 3 for theory details...
- Review the experimental situation...
- Zero-point energies are real...

Spontaneous emission

Spontaneous emission

- There is an argument, based on the Einstein A and B coefficients, that the zero-point energies simply have to be physically real in order to be consistent with the experimentally observed phenomenon of spontaneous emission....
- (Wikipedia says so, it must be true...)
- (Details mercifully suppressed for now...)

- Conclusion:
  For each 3-vector $\vec{k}$, and each polarization mode, the electromagnetic field is associated with a SHO of frequency $\omega = c\,|\vec{k}|$...
- So for n photons in this SHO one has

$$E_n = \left(n + \frac{1}{2}\right)\hbar\omega$$

- The total energy (per polarization) in the electromagnetic field is then

$$E_{tot} = \frac{V}{(2\pi)^3}\int d^3\vec{k}\,\left(n(\vec{k}) + \frac{1}{2}\right)\hbar\omega(\vec{k})$$

- Even in the quantum vacuum state one has

$$E_{tot} = \frac{V}{(2\pi)^3}\int d^3\vec{k}\,\frac{1}{2}\hbar\omega(\vec{k})$$

- But the integral

$$\int d^3\vec{k}\ \ \frac{1}{2}\hbar\omega(\vec{k}) = \int d^3\vec{k}\ \ \frac{1}{2}\hbar c|\vec{k}|$$

  diverges...
- WTF?
- We need to think about this...
- Carefully...

Zero-point energy density

Zero-point energy density:

- The zero-point energy density in the electromagnetic field is:

$$\rho_{zpe} = 2 \times \frac{\hbar c}{2} \int \frac{\mathrm{d}^3 k}{(2\pi)^3} \; |\vec{k}|.$$

- Introduce a cutoff:

$$\rho_{zpe} = 2 \times \frac{\hbar c}{2} \int_0^K \frac{\mathrm{d}^3 k}{(2\pi)^3} \; |\vec{k}| = \frac{\hbar c}{2\pi^2} \int_0^K dk \; k^3 = \frac{\hbar c}{8\pi^2} \; K^4$$

- Extremely naively,
(much better will be done tomorrow in Lecture 2),
let us assert that the cutoff is at the Planck scale...

Significance of the Planck scale

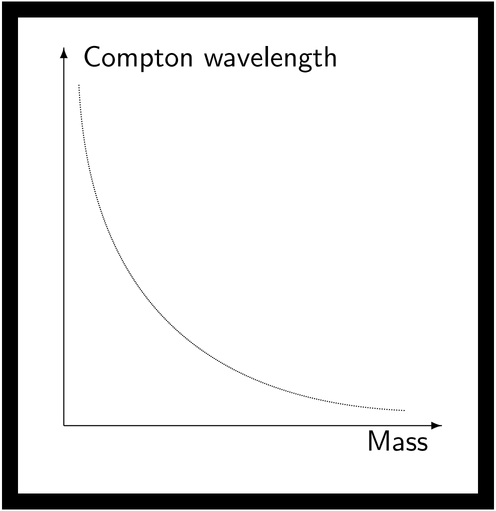
Significance of the Planck scale:

- Quantum mechanics tells us that an elementary particle of mass m can be reasonably easily localized within a distance

$$\lambda_{\text{Compton}} = \frac{\hbar}{mc}$$

known as the Compton wavelength.

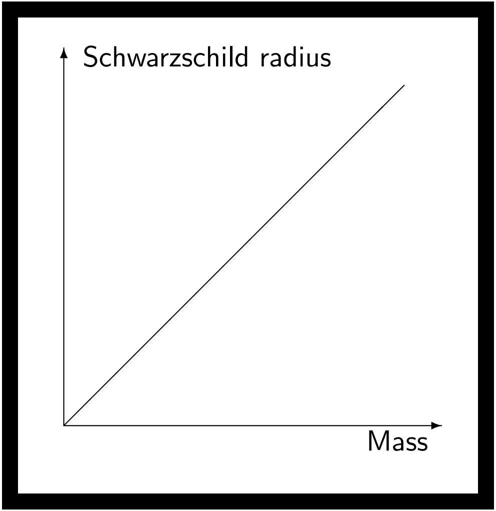
Significance of the Planck scale:



Significance of the Planck scale:

- Classical gravity tells us that a particle of mass m will disappear down a black hole if the particle is smaller than its Schwarzschild radius

$$r_{\text{Schwarzschild}} = \frac{2Gm}{c^2}.$$


Schwarzschild radius
Mass

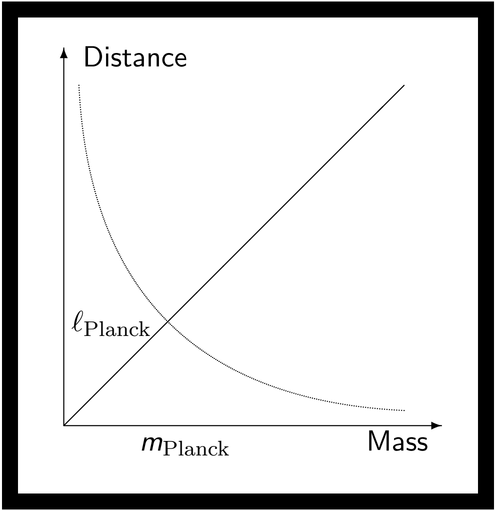
Significance of the Planck scale:

- A heavy enough elementary particle should disappear down its own little black hole.
- We expect this to happen when the Compton wavelength equals the Schwarzschild radius.
- Set $\lambda_{\text{Compton}} = r_{\text{Schwarzschild}} \cdots$
- Solve for the mass m of the particle...
- This defines the Planck mass:

$$M_{\text{Planck}} = \sqrt{\frac{\hbar c}{G}}.$$

- If we plot the Compton wavelength as a function of mass, and the Schwarzschild radius as a function of mass, the Planck mass is the place that the two graphs cross.

Significance of the Planck scale:



Significance of the Planck scale:

- Once we have the Planck mass, the Planck energy is easy: Take $E_{\mathrm{Planck}} = m_{\mathrm{Planck}} c^2$ to get

$$E_{\mathrm{Planck}} = \sqrt{\frac{\hbar c^5}{G}}.$$

- The Compton wavelength of a Planck mass particle, $\lambda_{\mathrm{Planck}} = \hbar/(m_{\mathrm{Planck}} c)$, is defined to be the Planck length

$$\ell_{\mathrm{Planck}} = \sqrt{\hbar c G}.$$

- Finally the Planck time is defined to be the time required for light to travel one Planck length, $T_{\mathrm{Planck}} = \ell_{\mathrm{Planck}}/c$, so that

$$T_{\mathrm{Planck}} = \sqrt{\frac{\hbar G}{c}}.$$

Significance of the Planck scale:

- From the theorists' perspective, one of the most frustrating aspects of our times is that all the interesting physics, (interesting from the point of view of quantum gravity that is), seems to be taking place at or above the Planck scale — but our current technology is simply not up to the task of building a Planck scale accelerator.
- From the way I defined the Planck scale above, it should be reasonably clear that the Planck regime is the border between classical physics (the Schwarzschild radius) and quantum physics (the Compton wavelength).

Significance of the Planck scale:

- Historically, Planck scale was first discussed by Max Planck in 1899.
- At the time quantum physics was in its infancy, the Planck constant just having been discovered as way of parameterizing the unexpected behaviour of black body radiation.
- Because of the then ill-understood nature of quantum physics, the Planck scale seemed at the time to be merely an accident of "algebraic numerology" — you put $\hbar$, c and G together in various ways and out popped masses, times, and distances.
- It is only after the development of quantum physics that the significance of the Planck scale as the harbinger of quantum gravity was appreciated.

Significance of the Planck scale:

The Planck scale.

Symbol	Name	Value
m_{Planck}	Planck mass	2.18×10^{-8} kilogram 21.8 micro-gramme 1.22×10^{19} GeV/c^2
E_{Planck}	Planck energy	1.22×10^{19} GeV
ℓ_{Planck}	Planck length	1.62×10^{-35} metres
T_{Planck}	Planck time	5.39×10^{-44} seconds

Significance of the Planck scale:

- Extremely naive suggestion:
- Cutoff the ZPE at Planck scale...

$$\rho_{zpe} = \frac{\hbar c}{8\pi^2} K^4 = \frac{\hbar c}{8\pi^2} \left(\frac{2\pi}{\ell_{\mathrm{Planck}}} \right)^4 = 2\pi^2 \frac{E_{\mathrm{Planck}}}{{\ell_{\mathrm{Planck}}}^3}.$$

- This is a ludicrously high energy density...

Worst prediction in particle physics?

Worst prediction in particle physics?

- The (extremely naive) estimate

$$\rho_{zpe} = 2\pi^2 \; \frac{E_{\text{Planck}}}{\ell_{\text{Planck}}{}^3},$$

  is about 10^{123} times higher than the observed average energy density in the universe...
- So the universe should curl up into a little ball 10^{-35} metres across?
- Infamously referred to as "the worst prediction in particle physics"...
- Actually it is a downright silly prediction...
- The real situation is much better than this extremely naive estimate...
- Details in Lecture 2 tomorrow...

Conclusions

Conclusions:

- Zero-point energies are objectively real...
- Zero-point energies are interesting...
- Zero-point energies are sometimes tricky...
- More tomorrow in Lecture 2...

End of Lecture 1.