Victoria University of Wellington

Te Whare Wānanga o te Ūpoko o te Ika a Maui

Lecture 3 of 4: Why are Casimir energy differences so often finite?

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Overview:

- One of the very first applications of the quantum vacuum was in the development of the notion of Casimir energy.
- Casimir energies, considered individually, are typically infinite.
- But differences in Casimir energies are often finite — a fortunate circumstance which luckily made some of the early calculations, (parallel plates and hollow spheres), tractable.
- Can this observation be systematized?
- What are necessary and sufficient conditions for Casimir energy differences to be finite?
- And when the Casimir energy differences are not finite, can anything useful be said?

- I shall argue (mathematically) that there are a large number of interesting physical situations where Casimir energy differences, (and so Casimir energy forces), are automatically known to be finite, even before starting specific computations.
- I shall argue (mathematically) that one can often develop physically interesting "reference models" such that the Casimir energy difference between the physical system and the "reference models" is known to be finite, even before starting specific computations.

Outline:

Introduction

Lemma

Exact result:

$$\omega - \omega_* = \frac{1}{\sqrt{4\pi}} \int_0^\infty \frac{\mathrm{d}t}{t^{3/2}} \left\{ e^{-\omega_*^2 t} - e^{-\omega^2 t} \right\}$$

Exact result:

$$\sum_n \{\omega_n - (\omega_*)_n\} = \frac{1}{\sqrt{4\pi}} \sum_n \int_0^\infty \frac{\mathrm{d}t}{t^{3/2}} \left\{ e^{-(\omega_*^2)_n t} - e^{-\omega_n^2 t} \right\}$$

Formally, (and I will justify this much more carefully later on):

$$\sum_n \{\omega_n - (\omega_*)_n\} = \frac{1}{\sqrt{4\pi}} \int_0^\infty \frac{\mathrm{d}t}{t^{3/2}} \sum_n \left\{ e^{-(\omega_*)_n^2 t} - e^{-\omega_n^2 t} \right\}$$

Then in terms of the heat kernel,

$$K(t) = \sum_n e^{-\omega_n^2 t},$$

we formally have:

$$\sum_n \{\omega_n - (\omega_*)_n\} = \int_0^\infty \frac{\mathrm{d}t}{t} \frac{1}{\sqrt{4\pi t}} \{K_*(t) - K(t)\}$$

- The heat kernel is an extremely useful object...
- The heat kernel is an commonly occurring object...

- The flat-space no-boundary diffusion operator in 3-dimensions:

$$\langle x | e^{t\nabla^2} | y \rangle = \frac{\exp\left(-\frac{[x-y]^2}{4t} \right)}{(4\pi t)^{3/2}}$$

- The flat-space no-boundary heat kernel in 3-dimensions:

$$K(t) = \langle x | e^{t\nabla^2} | x \rangle = \frac{1}{(4\pi t)^{3/2}}$$

- This generalizes in curved spacetime...
- This generalizes in the presence of boundaries...

- By the Seeley–de Witt expansion:

$$K(t) = (4\pi t)^{-d/2} \left\{ \sum_{i=0}^{N} a_{i/2}\; t^{i/2} + \mathcal{O}\left(t^{(N+1)/2} \right) \right\}$$

- Also:

$$K_*(t) = (4\pi t)^{-d/2} \left\{ \sum_{i=0}^{N} (a_*)_{i/2}\; t^{i/2} + \mathcal{O}\left(t^{(N+1)/2} \right) \right\}$$

- Don't panic — other people have done all the work for you...

Introduction:

- Seeley–de Witt is closely related to the density of states...
- As every schoolchild knows:

$$\text{(density of states)} = \frac{Vk^2}{2\pi^2} + ...$$

- As every schoolchild should know:

$$\text{(density of states)} = \frac{Vk^2}{2\pi^2} + \epsilon \; \frac{Sk}{8\pi} + \dots$$

where

$$\epsilon = \begin{cases} +1 & \text{for Neumann;} \\ 0 & \text{for periodic;} \\ -1 & \text{for Dirichlet.} \end{cases}$$

- This generalizes...
- Finite boundaries $\Longrightarrow$ modified density of states...

Introduction

In terms of the heat kernel we formally have:

$$\sum_n \{\omega_n - (\omega_*)_n\} = \int_0^\infty \frac{\mathrm{d}t}{t} \frac{1}{\sqrt{4\pi t}} \{K_*(t) - K(t)\}$$

and

$$K_*(t) - K(t) = (4\pi t)^{-d/2} \left\{ \sum_{i=0}^{N} \left\{ (a_*)_{i/2} - a_{i/2} \right\} \; t^{i/2} + \mathcal{O}\left(t^{(N+1)/2} \right) \right\}$$

Now choose $N = d + 1$, then formally

$$\sum_n \{\omega_n - (\omega_*)_n\} = \int_0^\infty \frac{\mathrm{d}t}{t} (4\pi t)^{-(d+1)/2} \left\{ \sum_{i=0}^{d+1} \left\{ (a_*)_{i/2} - a_{i/2} \right\} \; t^{i/2} \right\}$$

$$+(\text{UV finite})$$

That is:

Lemma (At this stage a formal argument only)

$$\Delta(\textit{Casimir Energy}) = -\frac{\hbar}{2}\int_0^\infty \frac{\mathrm{d}t}{t}(4\pi t)^{-(d+1)/2}\left\{\sum_{i=0}^{d+1}\Delta a_{i/2}\; t^{i/2}\right\}$$

$$+(\textit{UV finite})$$

The rest of the lecture will involve refinements on this simple theme...

- In 3+1 dimensions want $\Delta a_0 = \Delta a_{1/2} = \Delta a_1 = \Delta a_{3/2} = \Delta a_2 = 0$.
- In 2+1 dimensions want $\Delta a_0 = \Delta a_{1/2} = \Delta a_1 = \Delta a_{3/2} = 0$.
- In 1+1 dimensions want $\Delta a_0 = \Delta a_{1/2} = \Delta a_1 = 0$.

Being more careful

Let's regulate everything a little more carefully....

Lemma

Exact result:

$$\omega\ \mathrm{erfc}(\omega/\Omega) = \frac{\Omega}{\sqrt{\pi}} e^{-\omega^2/\Omega^2} - \frac{1}{\sqrt{4\pi}} \int_{\Omega^{-2}}^{\infty} \frac{\mathrm{d}t}{t^{3/2}} e^{-\omega^2 t}$$

Exact result:

$$\sum_n \omega_n\ \mathrm{erfc}(\omega_n/\Omega) = \frac{\Omega}{\sqrt{\pi}} \sum_n e^{-\omega_n^2/\Omega^2} - \frac{1}{\sqrt{4\pi}} \sum_n \int_{\Omega^{-2}}^{\infty} \frac{\mathrm{d}t}{t^{3/2}} e^{-\omega_n^2 t}$$

Exact result (no longer just formal):

$$\sum_n \omega_n\ \mathrm{erfc}(\omega_n/\Omega) = \frac{\Omega}{\sqrt{\pi}} \sum_n e^{-\omega_n^2/\Omega^2} - \frac{1}{\sqrt{4\pi}} \int_{\Omega^{-2}}^{\infty} \frac{\mathrm{d}t}{t^{3/2}} \sum_n e^{-\omega_n^2 t}$$

Being more careful

In terms of the heat kernel:

$$\sum_n \omega_n \, \mathrm{erfc}(\omega_n/\Omega) = \frac{\Omega}{\sqrt{\pi}} K(\Omega^{-2}) - \frac{1}{\sqrt{4\pi}} \int_{\Omega^{-2}}^{\infty} \frac{\mathrm{d}t}{t^{3/2}} K(t)$$

Now apply the Seeley–de Witt expansion:

$$K(t) = (4\pi t)^{-d/2} \left\{ \sum_{i=0}^{N} a_{i/2} \; t^{i/2} + \mathcal{O}\left(t^{(N+1)/2} \right) \right\}$$

Being more careful

But then (choose $N = d$):

$$\frac{\Omega}{\sqrt{\pi}} K(\Omega^{-2}) = 2\left(\frac{\Omega}{\sqrt{4\pi}}\right)^{d+1} \left\{ \sum_{i=0}^{d} a_{i/2}\, \Omega^{-i} \right\} + (\text{finite as } \Omega \to \infty)$$

The integral is a little trickier...

$$\frac{1}{\sqrt{4\pi}} \int_{\Omega^{-2}}^{\infty} \frac{\mathrm{d}t}{t^{3/2}} K(t) = \int_{\Omega^{-2}}^{\infty} \frac{\mathrm{d}t}{t} \frac{1}{\sqrt{4\pi t}} K(t)$$

In this integral choose $N = d + 1$.

Then, treating the logarithmic term separately, we have

$$\int_{\Omega^{-2}}^{\infty} \frac{\mathrm{d}t}{t} \frac{1}{\sqrt{4\pi t}} K(t) = \frac{1}{\sqrt{4\pi}} \int_{\Omega^{-2}}^{\infty} \frac{\mathrm{d}t}{t^{3/2}} (4\pi t)^{-d/2} \left\{ \sum_{i=0}^{d} \left\{ a_{i/2} \right\} \; t^{i/2} \right\}$$

$$+ \frac{a_{(d+1)/2}}{(4\pi)^{(d+1)/2}} \ln(\Omega^2) + (\text{finite as } \Omega \to \infty)$$

Combine:

$$\int_{\Omega^{-2}}^{\infty} \frac{\mathrm{d}t}{t} \frac{1}{\sqrt{4\pi t}} K(t) = \int_{\Omega^{-2}}^{\infty} \frac{\mathrm{d}t}{t} (4\pi t)^{-(d+1)/2} \left\{ \sum_{i=0}^{d} \left\{ a_{i/2} \right\} \; t^{i/2} \right\}$$

$$+ \frac{a_{(d+1)/2}}{(4\pi)^{(d+1)/2}} \ln(\Omega^2) + (\text{finite as } \Omega \to \infty)$$

Being more careful

Performing the remaining integrals:

$$\int_{\Omega^{-2}}^{\infty} \frac{\mathrm{d}t}{t} \frac{1}{\sqrt{4\pi t}} K(t) = -\frac{1}{(4\pi)^{(d+1)/2}} \left\{ \sum_{i=0}^{d} \frac{a_{i/2}\, \Omega^{d+1-i}}{d+1-i} \right\}$$

$$+ \frac{a_{(d+1)/2}}{(4\pi)^{(d+1)/2}} \ln(\Omega^2) + (\text{finite as } \Omega \to \infty)$$

Assembling all the pieces:

$$\sum_n \omega_n \,\mathrm{erfc}(\omega_n/\Omega) = 2\left(\frac{\Omega}{\sqrt{4\pi}}\right)^{d+1}\left\{\sum_{i=0}^{d}\left\{a_{i/2}\right\}\ \Omega^{-i}\right\}$$

$$+\frac{1}{(4\pi)^{(d+1)/2}}\left\{\sum_{i=0}^{d}\frac{a_{i/2}\,\Omega^{d+1-i}}{d+1-i}\right\}+\frac{a_{(d+1)/2}}{(4\pi)^{(d+1)/2}}\ln(\Omega^2)$$

$$+(\text{finite as } \Omega\to\infty)$$

Being more careful

Lemma

We have:

$$\sum_n \omega_n \,\mathrm{erfc}(\omega_n/\Omega) = \left\{ \sum_{i=0}^{d} k_i \; a_{i/2} \; \Omega^{d+1-i} \right\} + k_{(d+1)/2} \; a_{(d+1)/2} \; \ln(\Omega^2)$$

$$+(\textit{finite as } \Omega \to \infty)$$

For our purposes the specific values of the k_i are not important...

Regulated Casimir Energy

Regulated Casimir Energy

Consider the regulated Casimir energy:

$$\text{(Regulated Casimir energy)} = \frac{1}{2}\hbar \sum_n \omega_n \, \mathrm{erfc}(\omega_n/\Omega)$$

Then:

Theorem

(Regulated Casimir energy) $=$

$$\frac{1}{2}\hbar \left\{ \sum_{i=0}^{d} k_i \; a_{i/2} \; \Omega^{d+1-i} \right\} + \frac{1}{2}\hbar k_{(d+1)/2} \; a_{(d+1)/2} \; \ln(\Omega^2)$$

$$+(\textit{finite as } \Omega \to \infty)$$

Take differences:

Δ(Regulated Casimir energy) $=$

$$\frac{1}{2}\hbar \left\{ \sum_{i=0}^{d} k_i \; \Delta a_{i/2} \; \Omega^{d+1-i} \right\} + \frac{1}{2}\hbar k_{(d+1)/2} \; \Delta a_{(d+1)/2} \; \ln(\Omega^2)$$

$$+(\text{finite as } \Omega \to \infty)$$

If the first $(d+1)/2$ Seeley–de Witt coefficients are unchanged,

$$\Delta a_0 = \Delta a_{1/2} = \dots = \Delta a_{(d+1)/2} = 0,$$

then:

$$\Delta(\text{Regulated Casimir energy}) = (\text{finite as } \Omega \to \infty)$$

We can now safely take the limit...

Regulated Casimir Energy

In the limit where the cutoff is removed ($\Omega \to \infty$) we have:

Theorem

When comparing two physical situations where the first $\frac{(d+1)}{2}$ Seeley–de Witt coefficients are the same,

$$\Delta a_0 = \Delta a_{1/2} = \cdots = \Delta a_{(d+1)/2} = 0,$$

we have:

$$\Delta(\textit{Casimir energy}) = (\textit{finite}).$$

How general is this phenomenon?

Unchanging Seeley–de Witt coefficients

Unchanging Seeley–de Witt coefficients:

There are very many physically interesting situations where the Seeley–de Witt coefficients are unchanging...

The pre-eminent cases are these:

- Parallel plates.
- Thin spherical shells.

In both cases an infra-red regulator is needed,
and some subtle thought is required...
More radically:

- Take any collection of conductors.
- Move them around relative to each other.
(Without distorting their shapes and/or volumes.)
- Then the change in Casimir energy is finite.
- Then the Casimir forces are finite.

For a region **V** with boundary ∂**V**:

$$a_0 \propto \int_{\mathbf{V}} 1 \; \mathrm{d}^d x = \text{(volume)}$$

$$a_{1/2} \propto \int_{\partial\mathbf{V}} 1 \; \mathrm{d}^{d-1} x = \text{(surface area)}$$

$$a_1 \propto \int_{\mathbf{V}} \{R, V\} \; \mathrm{d}^d x + \int_{\partial\mathbf{V}} \{K\} \; \mathrm{d}^{d-1} x$$

$$a_{3/2} \propto \int_{\partial\mathbf{V}} \{R, V, K^2, K_{ij}K^{ij}\} \; \mathrm{d}^{d-1} x$$

$$a_2 \propto \int_{\mathbf{V}} \{...\} \; \mathrm{d}^d x + \int_{\partial\mathbf{V}} \{...\} \; \mathrm{d}^{d-1} x$$

Here the $\{\_, \_, \_\}$ denote species-dependent linear combinations...

Unchanging Seeley–de Witt coefficients:

In all its glory:

$$
\begin{aligned}
a_2 \quad \propto \quad & \int_{\mathbf{V}} \{R^2, V^2, RV, \nabla^2 R, \nabla^2 V, R_{ab}R^{ab}, R_{abcd}R^{abcd}\} \, \mathrm{d}^d x \\
& + \int_{\partial \mathbf{V}} \{R_{;n}, V_{;n}, K_{ii:jj}, K_{ij:ij}, VK, K^3, \mathrm{tr}(K^2)K, \mathrm{tr}(K^3)\} \, \mathrm{d}^{d-1} x \\
& + \int_{\partial \mathbf{V}} \{RK, g^{ij} R_{ninj} K, R_{ninj} K^{ij}, g^{ik} R_{ijkl} K^{jl}\} \, \mathrm{d}^{d-1} x
\end{aligned}
$$

Here the $\{\_,\_,\_\}$ denote species-dependent linear combinations...

(There are also contributions from kinks and corners;
but let's stay with smooth boundaries for now.)

Unchanging Seeley–de Witt coefficients:

Parallel plates:

Working with QED in flat spacetime with flat boundaries:

$$
\begin{aligned}
a_0 &\propto \text{(volume)} \\
a_{1/2} &\propto \text{(surface area)} \\
a_1 &= 0 \\
a_{3/2} &= 0 \\
a_2 &= 0
\end{aligned}
$$

Just keep volume and surface area fixed...

For example:
Periodic boundary conditions in $d-1$ directions...
Conducting box boundary conditions in the remaining direction...

Unchanging Seeley–de Witt coefficients:

Hollow spheres:

Working with QED in flat spacetime with thin spherical boundaries:

Step I (QED in flat spacetime):

$$
\begin{aligned}
a_0 &\propto \text{(volume)} \\
a_{1/2} &\propto \text{(surface area)} \\
a_1 &\propto \int_{\partial \mathbf{V}} \{K\} \, \mathrm{d}^{d-1}x \\
a_{3/2} &\propto \int_{\partial \mathbf{V}} \{K^2, K_{ij}K^{ij}\} \, \mathrm{d}^{d-1}x \\
a_2 &\propto \int_{\partial \mathbf{V}} \{g^{ij}g^{kl}K_{ij:kl}, K^{ij}{}_{:ij}, K^3, \mathrm{tr}(K^2)K, \mathrm{tr}(K^3)\} \, \mathrm{d}^{d-1}x
\end{aligned}
$$

Now a little trickier...

Unchanging Seeley–de Witt coefficients:

Hollow spheres:

Working with QED in flat spacetime with thin spherical boundaries:

Step II:
As long as the boundaries are thin, then $K_{\mathrm{inside}} = -K_{\mathrm{outside}}$, leading to cancellations in a_1 and a_2.
(The outermost boundary, the IR regulator, is always held fixed.)
Then:

$$
\begin{aligned}
\Delta a_0 &\to 0 \\
\Delta a_{1/2} &\propto \Delta(\text{surface area}) \\
\Delta a_1 &\to 0 \\
\Delta a_{3/2} &\propto \int_{\partial \mathbf{V}} \{K^2, K_{ij}K^{ij}\}\, \mathrm{d}^{d-1}x \\
\Delta a_2 &\to 0
\end{aligned}
$$

Hollow spheres:

Working with QED in flat spacetime with thin spherical boundaries:

Step III:
As long as the inner boundaries are simply rescaled,
then $\int_{\partial \mathbf{V}} KK \mathrm{d}^2 x$ is scale invariant, leading to a cancellation in $a_{3/2}$.
(The outermost boundary, the IR regulator, is always held fixed.)
Then:

$$
\begin{aligned}
\Delta a_0 &\to 0 \\
\Delta a_{1/2} &\propto \Delta(\text{surface area}) \\
\Delta a_1 &\to 0 \\
\Delta a_{3/2} &\to 0 \\
\Delta a_2 &\to 0
\end{aligned}
$$

Hollow spheres:

Working with QED in flat spacetime with thin spherical boundaries:

Step IV:
In spherical symmetry, define TE and TM modes.
Note that they have equal and opposite contributions to $a_{1/2}$,
leading to a cancellation in $a_{1/2}$.
(The outermost boundary is always held fixed.)
Then:

$$\begin{aligned} \Delta a_0 &\to 0 \\ \Delta a_{1/2} &\to 0 \\ \Delta a_1 &\to 0 \\ \Delta a_{3/2} &\to 0 \\ \Delta a_2 &\to 0 \end{aligned}$$

Unchanging Seeley–de Witt coefficients:

Hollow spheres:

Working with QED in flat spacetime with thin spherical boundaries:

$$\Delta(\text{Casimir Energy}) = (\text{finite})$$

This underlies the "miraculous cancellations" in Boyer's calculation of the Casimir energy of a hollow sphere.

Compare two hollow spheres of radius a and b;
letting the IR regulator move out to infinity:

$$\Delta(\text{Casimir Energy}) = \hbar\, c\; B \left(\frac{1}{a} - \frac{1}{b} \right)$$

"All" one needs to do is to calculate the numerical coefficient B, which is now guaranteed to be finite...

Unchanging Seeley–de Witt coefficients:

If one has determined

$$\Delta(\text{Casimir Energy}) = (\text{finite})$$

then

$$\Delta(\text{Casimir Energy}) = \frac{1}{2}\hbar \left\{\text{any resummation technique}\right\} (\omega_n - (\omega_*)_n)$$

Boyer uses Riesz resummation.

This is justified only in hindsight...

Blindly calculating

$$\sum_n (\omega_n - (\omega_*)_n)$$

is asking for trouble...

Unchanging Seeley–de Witt coefficients:

Generalize — Working with QED in flat spacetime:

$$
\begin{aligned}
a_0 &\propto \text{(volume)} \\
a_{1/2} &\propto \text{(surface area)} \\
a_1 &\propto \int_{\partial \mathbf{V}} \{K\} \, \mathrm{d}^{d-1}x \\
a_{3/2} &\propto \int_{\partial \mathbf{V}} \{K^2, K_{ij}K^{ij}\} \, \mathrm{d}^{d-1}x \\
a_2 &\propto \int_{\partial \mathbf{V}} \{g^{ij}g^{kl}K_{ij:kl}, K^{ij}{}_{:ij}, K^3, \mathrm{tr}(K^2)K, \mathrm{tr}(K^3)\} \, \mathrm{d}^{d-1}x
\end{aligned}
$$

- Take any collection of conductors.
- Move them around relative to each other.
  (Without distorting their shapes and/or volumes.)
- Then the change in Casimir energy is finite.
- Then the Casimir forces are finite.

Unchanging Seeley–de Witt coefficients:

Working in flat spacetime with periodic boundary conditions:

We have:

$$
\begin{aligned}
a_0 &\propto \text{(volume)} \\
a_{1/2} &= 0 \\
a_1 &\propto \int_{\mathbf{V}} \{V\}\, \mathrm{d}^d x \\
a_{3/2} &= 0 \\
a_2 &\propto \int_{\mathbf{V}} \{V^2\}\, \mathrm{d}^d x
\end{aligned}
$$

We "just" need to keep a_0, a_1, and a_2 fixed...

Unchanging Seeley–de Witt coefficients:

Working in flat spacetime with periodic boundary conditions:

In (1+1) dimensions define

$$\overline{V} = \frac{\int_0^L V \,\mathrm{d}x}{L}$$

Compare the two situations:

- $D = \nabla^2 + V(x)$; eigenvalues ω_n^2.
- $\overline{D} = \nabla^2 + \overline{V}$; eigenvalues $\overline{\omega}_n^2$.

Then

$$\sum_n \{\omega_n \,\mathrm{erfc}(\omega_n/\Omega) - \overline{\omega}_n \,\mathrm{erfc}(\overline{\omega}_n/\Omega)\} = (\text{finite as } \Omega \to \infty)$$

$$(\text{Casimir energy of } D) - (\text{Casimir energy of } \overline{D}) = (\text{finite})$$

Working in flat spacetime with periodic boundary conditions:

In (3+1) dimensions define

$$\overline{V} = \frac{\int_0^{\mathbf{V}} V(x)\, \mathrm{d}^3 x}{\text{volume}(\mathbf{V})}; \qquad \overline{V^2} = \frac{\int_0^{\mathbf{V}} V(x)^2\, \mathrm{d}^3 x}{\text{volume}(\mathbf{V})};$$

Now solve

$$m_1^2 + m_2^2 = 2\overline{V}; \qquad m_1^4 + m_2^4 = 2\overline{V^2}$$

Compare the three situations:

- $D = \nabla^2 + V(x)$; eigenvalues ω_n^2.
- $\overline{D_1} = \nabla^2 + m_1^2$; eigenvalues $\overline{(\omega_1)}_n^2$.
- $\overline{D_2} = \nabla^2 + m_2^2$; eigenvalues $\overline{(\omega_2)}_n^2$.

Unchanging Seeley–de Witt coefficients:

Working in flat spacetime with periodic boundary conditions:

Then:

$$\sum_n \left\{ \omega_n \, \mathrm{erfc}(\omega_n/\Omega) - \frac{1}{2}\, \overline{(\omega_1)}_n \, \mathrm{erfc}\left(\overline{(\omega_1)}_n/\Omega \right) - \frac{1}{2}\, \overline{(\omega_2)}_n \, \mathrm{erfc}\left(\overline{(\omega_2)}_n/\Omega \right) \right\}$$

$$= (\text{finite as } \Omega \to \infty)$$

This implies:

$$(\text{Casimir energy of } D) - \frac{1}{2}\left(\text{Casimir energy of } \overline{D_1}\right) - \frac{1}{2}\left(\text{Casimir energy of } \overline{D_2}\right)$$

$$= (\text{finite})$$

And if the Casimir energy differences are not finite?

If the Casimir energy differences are not finite?

- Real metals and real dielectrics are transparent in the UV.
- The UV cutoff Ω is a stand-in for all the complicated physics.
- Let us write a general cutoff function as

$$f\left(\frac{\omega}{\Omega}\right) = \int_0^\infty g(\xi)\,\mathrm{erfc}\left(\frac{\omega}{\xi\Omega}\right)\,\mathrm{d}\xi; \qquad \int_0^\infty g(\xi)\,\mathrm{d}\xi = 1.$$

- Note $f(0) = 1$, while $f(\infty) = 0$, and f is monotone decreasing.
- Let us now consider

$$\sum_n \omega_n\, f\left(\frac{\omega_n}{\Omega}\right)$$

If the Casimir energy differences are not finite?

Then

$$\sum_n \omega_n \,\mathrm{erfc}(\omega_n/\Omega) = \left\{ \sum_{i=0}^{d} k_i \; a_{i/2} \; \Omega^{d+1-i} \right\} + k_{(d+1)/2} \; a_{(d+1)/2} \; \ln(\Omega^2)$$

$$+(\text{finite as } \Omega \to \infty)$$

becomes

$$\sum_n \omega_n \; f\left(\frac{\omega_n}{\Omega}\right) = \left\{ \sum_{i=0}^{d} k_i \left(\int_0^\infty g(\xi)\; \xi^{d+1-i} \;\mathrm{d}\xi \right) \; a_{i/2} \; \Omega^{d+1-i} \right\}$$

$$+k_{(d+1)/2} \; a_{(d+1)/2} \; \left\{ \ln(\Omega^2) + 2 \int_0^\infty g(\xi) \; \ln\xi \;\mathrm{d}\xi \right\}$$

$$+(\text{finite as } \Omega \to \infty)$$

If the Casimir energy differences are not finite?

Theorem

- *For a general cutoff $f(\omega/\Omega)$ one has*

$$\sum_n \omega_n \, f\left(\frac{\omega_n}{\Omega}\right) = \left\{ \sum_{i=0}^{d} [k(f)]_i \;\; a_{i/2} \;\; \Omega^{d+1-i} \right\}$$

$$+k_{(d+1)/2} \;\; a_{(d+1)/2} \;\; \ln(\Omega^2)$$

$$+(\textit{finite as } \Omega \to \infty)$$

- *The $[k(f)]_i$ are phenomenological parameters that depend on the detailed physics of the specific cutoff function $f(\omega/\Omega)$.*
- *However $k_{(d+1)/2}$ is cutoff independent.*
- *The Ω dependence represents real physics.*
- *Live with it!*

If the Casimir energy differences are not finite?

Definition

- For a general cutoff $f(\omega/\Omega)$ one has

$$\Delta(\text{Casimir energies}) \;=\; \frac{1}{2}\hbar\,\Delta\left(\sum_n \omega_n\, f\left(\frac{\omega_n}{\Omega}\right)\right)$$

If the Casimir energy differences are not finite?

Theorem

- *For a general cutoff $f(\omega/\Omega)$ one has*

$$\begin{aligned}\Delta(\textit{Casimir energies}) &= \frac{1}{2}\hbar\left\{\sum_{i=0}^{d}[k(f)]_i\ \Delta a_{i/2}\ \Omega^{d+1-i}\right\} \\ &\quad +\frac{1}{2}\hbar\ k_{(d+1)/2}\ \Delta a_{(d+1)/2}\ \ln(\Omega^2) \\ &\quad +(\textit{finite as } \Omega \to \infty)\end{aligned}$$

- *The $[k(f)]_i$ are phenomenological parameters that depend on the detailed physics of the specific cutoff function $f(\omega/\Omega)$.*
- *However $k_{(d+1)/2}$ is cutoff independent.*
- *The Ω dependence represents real physics. Live with it!*

Conclusions

Conclusions:

- In $(d+1)$ dimensions, iff the first few Seeley–de Witt coefficients agree,

$$\Delta a_0 = \Delta a_{1/2} = \dots \Delta a_{(d+1)/2} = 0,$$

  then the difference in Casimir energies is guaranteed finite.
- This is a useful thing to check before you start calculating.
- The erfc function, in the form $\mathrm{erfc}(\omega/\Omega)$, is a perhaps unexpectedly useful regulator

$$\mathrm{erfc}(0) = 1; \qquad \mathrm{erfc}(\infty) = 0$$

- Various generalizations, (such as counting eigenstates, or calculating sums of powers of eigenvalues), are also possible.

End of Lecture 3.