Victoria University of Wellington

Te Whare Wānanga o te Ūpoko o te Ika a Maui

Lecture 4 of 4: Sakharov-style induced gravity

Matt Visser

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Outline:

- Sakharov's 1967 notion of "induced gravity" continues to attract attention and periodically enjoys a bit of a resurgence.
- The basic idea, originally presented in a very brief 3-page paper, with a total of only 4 formulas, is that gravity is not "fundamental".
- Instead it was argued that gravity (general relativity) emerges from quantum field theory in roughly the same sense that hydrodynamics or continuum elasticity theory emerges from molecular physics.
- This resonates with current ideas on building "analogue spacetimes" for mimicking aspects of general relativity.
- While it is not possible (as yet) to get everything to work just right, there are definitely some intriguing hints as to how to proceed.

Outline:

Introduction

Introduction:

Einstein gravity (general relativity) is based on two things:

- pseudo-Riemannian geometry
  (Lorentzian geometry).
- field equations for the Ricci tensor.

So one naturally wonders:

- Q1: Are there other physical systems that naturally lead to the notion of Lorentzian geometry?
  (Analogue spacetimes.)
- Q2: Are there other physical systems that naturally lead to the Einstein equations involving the Ricci tensor?
  (Sakharov "induced gravity".)

Step 1: From Lagrangian to pseudo-Riemannian geometry.

- Q: How easy is it to get a notion of pseudo-Riemannian geometry from more primitive concepts?
- A: Very easy. (Surprisingly so).
- Need:
  - Lagrangian;
  - linearization;
  - hyperbolic PDEs.

(Details deferred.)

Lorentzian geometries show up in many a priori unexpected places.

Step 2: Inducing Einstein–Hilbert dynamics.

This is where Sakharov comes in:

- Assume you have a Lorentzian manifold.
- Make no assumptions about the dynamics of this geometry...
- Leave the geometry free to flap in the breeze...
- Do one-loop quantum field theory on this manifold.
- One-loop effective action guaranteed to contain terms of the form:

$$\int \mathrm{d}^4 x \sqrt{-g} \left\{ c_0\, \kappa^4 + c_1\, \kappa^2\, R + c_2\, (\text{``}R^2\text{''}) \right\}.$$

- This is suggestive, but is this enough?

$$\int \mathrm{d}^4 x \sqrt{-g} \left\{ -\Lambda - \frac{R}{16\pi\, G_N} + K\, (\text{``}R^2\text{''}) \right\}.$$

Einstein–Hilbert dynamics

One-loop effective action (scalar field):

$$\begin{aligned}\mathcal{S}_g &= -\frac{1}{2}\ln\det(\Delta_g + m^2 + \xi R)\\ &= -\frac{1}{2}\mathrm{Tr}\ln(\Delta_g + m^2 + \xi R).\end{aligned}$$

- One-loop diagrams with external gravitons.
- Notation:

$$\mathrm{Tr}[\ \ ] \equiv \int \mathrm{d}^4x\ \mathrm{tr}[\ \ ].$$

Einstein–Hilbert dynamics:

Use Frullani's integral:

$$\ln(b/a) = \int_0^\infty \frac{\mathrm{d}x}{x} \left[e^{-ax} - e^{-bx} \right].$$

Use Schwinger proper time formalism:

$$\mathcal{S}_g = \mathcal{S}_{g_0} + \frac{1}{2}\mathrm{Tr} \int_0^\infty \frac{\mathrm{d}s}{s} \left[\exp(-s[\Delta_g + m^2 + \xi R]) - \exp(-s[\Delta_{g_0} + m^2 + \xi R_0]) \right].$$

Einstein–Hilbert dynamics:

Regularize:

$$\mathcal{S}_g = \mathcal{S}_{g_0} + \frac{1}{2}\mathrm{Tr}\int_{\kappa^{-2}}^{\infty} \frac{\mathrm{d}s}{s} \left[\exp(-s[\Delta_g + m^2 + \xi R]) - \exp(-s[\Delta_{g_0} + m^2 + \xi R_0])\right].$$

Use heat kernel expansion as $s \to 0$:

$$\exp(-s[\Delta_g + m^2 + \xi R]) = \frac{1}{(4\pi\, s)^2}\left[a_0(g) + a_1(g)s + a_2(g)s^2 + \cdots\right].$$

— No boundary terms; bulk terms only...
— Four dimensions, Wick rotated Euclidean-signature...

Then

$$\mathcal{S}_g = \mathcal{S}_{g_0} + \frac{1}{32\pi^2}\mathrm{Tr}\Big\{ [a_0(g) - a_0(g_0)]\ \kappa^4/2$$
$$+ [a_1(g) - a_1(g_0)]\ \kappa^2$$
$$+ [a_2(g) - a_2(g_0)]\ \ln(\kappa^2/m^2) \Big\}$$
$$+ \text{UV finite.}$$

Dirac particles (no [net] chiral anomaly):

$$\begin{aligned}\mathcal{S} &= +\ln\det([\gamma\cdot D]+m)\\ &= +\frac{1}{2}\ln\det(-[\gamma\cdot D]^2+m^2)\end{aligned}$$

Note relative minus sign.

Einstein–Hilbert dynamics:

Summing over all particles, bose plus fermi:

$$\begin{aligned} \mathcal{S}_g = \mathcal{S}_{g_0} \quad &+ \quad \frac{1}{32\pi^2}\mathrm{Str}\Bigg\{ [a_0(g) - a_0(g_0)]\ \kappa^4/2 \\ &+ \quad [a_1(g) - a_1(g_0)]\ \kappa^2 \\ &+ \quad [a_2(g) - a_2(g_0)]\ \ln(\kappa^2/m^2) \Bigg\} \\ &+ \quad \text{UV finite.} \end{aligned}$$

The "supertrace" Str, weights fermi fields with a relative minus sign.

$$\mathrm{Str}[\ \ ] \equiv \mathrm{Tr}\left[(-)^F\ \ \right].$$

("Str" does not imply supersymmetry.)

Some mathematics/dimensional analysis:

$$a_0(g) = \sqrt{-g}.$$

$$a_1(g) = \sqrt{-g}\,\{k_1\ R(g) - m^2\}.$$

$$a_2(g) = \sqrt{-g}\left\{k_2\ C_{abcd}C^{abcd} + k_3\ R_{ab}R^{ab} + k_4\ R^2 \right.$$
$$\left. + k_5\ \nabla^2 R - m^2\ k_1\ R(g) + \frac{1}{2}m^4\right\}.$$

Upon integration:

$$\int a_2(g) = \int \sqrt{-g}\left\{k_2'\ C_{abcd}\ C^{abcd} + k_4'\ R^2 - m^2 k_1\ R(g) + \frac{1}{2}m^4\right\}.$$

Einstein–Hilbert dynamics:

Define:

$$\text{Str}[\ \ ] \equiv \int d^4x \; \text{str}[\ \ ].$$

Unwrap:

$$\begin{aligned}
\mathcal{S}_g = \mathcal{S}_{g_0} \quad &+ \quad \frac{1}{32\pi^2}\Bigg\{ \text{str}\left[\frac{\kappa^4}{2} - m^2\kappa^2 + \frac{m^4}{2}\ln\left(\frac{\kappa^2}{m^2}\right)\right] \int \mathrm{d}^4x \left[\sqrt{-g} - \sqrt{-g_0}\right] \\
&+ \quad \text{str}\left[k_1\kappa^2 - k_1 m^2 \ln\left(\frac{\kappa^2}{m^2}\right)\right] \int \mathrm{d}^4x \left[\sqrt{-g}\; R(g) - \sqrt{-g_0}\; R(g_0)\right] \\
&+ \quad \text{str}[k_2' \ln\left(\kappa^2/m^2\right)] \int \mathrm{d}^4x \left[\sqrt{-g}\; C^2(g) - \sqrt{-g_0}\; C^2(g_0)\right] \\
&+ \quad \text{str}[k_4' \ln\left(\kappa^2/m^2\right)] \int \mathrm{d}^4x \left[\sqrt{-g}\; R^2(g) - \sqrt{-g_0}\; R^2(g_0)\right] \Bigg\} \\
&+ \quad \text{UV finite.}
\end{aligned}$$

This is getting interesting...

Einstein–Hilbert dynamics:

Extract coefficients:

$$\Lambda = \Lambda_0 - \frac{1}{32\pi^2}\text{str}\left[\frac{\kappa^4}{2} - m^2\kappa^2 + \frac{m^4}{2}\ln\left(\frac{\kappa^2}{m^2}\right)\right] + \text{UV finite.}$$

$$\frac{1}{G} = \frac{1}{G_0} - \frac{1}{2\pi}\text{str}\left[k_1\kappa^2 - k_1 m^2 \ln\left(\frac{\kappa^2}{m^2}\right)\right] + \text{UV finite.}$$

$$K_2 = (K_2)_0 + \frac{1}{32\pi^2}\text{str}[k_2' \ln(\kappa^2/m^2)] + \text{UV finite.}$$

$$K_4 = (K_4)_0 + \frac{1}{32\pi^2}\text{str}[k_4' \ln(\kappa^2/m^2)] + \text{UV finite.}$$

Here the road diverges:

- Pauli (weak): Demand one-loop finiteness.
- Pauli (strong): Demand all-loop finiteness.
- Sakharov: Demand one-loop dominance.

Pauli: one-loop finiteness

Pauli: one-loop finiteness (compensation):

(Pauli was working in flat space.)
To guarantee a one-loop finite result need:

$$\mathrm{str}(I) = \mathrm{str}(m^2) = \mathrm{str}(m^4) = 0.$$

(Pauli Lectures on Physics, V6, p33, 1950-51)
Then:

$$\Lambda = \Lambda_0 + \frac{1}{64\pi^2}\,\mathrm{str}\left[m^4 \ln\left(\frac{m^2}{\mu^2}\right)\right] + \text{two loops.}$$

(known to Pauli, at least implicitly.)
This is a fore-runner of supersymmetry, certainly known to Bruno Zumino.
It gives tight constraints on the particle physics content of the model;
constraints that are certainly not satisfied by the standard model.
(These constraints are satisfied by SUSY theories before SSB;
and by all of the non-SUSY finite QFTs.)

Extending Pauli's idea to curved space (variant of Frolov & Furasev):

$$\mathrm{str}(k_1) = \mathrm{str}(k_1\ m^2) = 0.$$

Here:

$$\begin{aligned}
k_1(s=0) &= \frac{1}{6} - \xi; \\
k_1(s=\tfrac{1}{2}) &= -\frac{1}{6}; \qquad \text{(Weyl)} \\
k_1(s=\tfrac{1}{2}) &= -\frac{1}{3}; \qquad \text{(Dirac)} \\
k_1(s=1) &= -\frac{2}{3}. \qquad \text{(Photon)}
\end{aligned}$$

These are now very strong constraints on the particle content.
(These constraints are not derivable from SUSY alone.)

Pauli: one-loop finiteness (compensation):

- Terra incognita.
- N=1 SUGRA does force $\xi = 0$.
- Softly broken N=2 SUGRA?
- The Newton constant is:

$$\frac{1}{G} = \frac{1}{G_0} - \frac{1}{2\pi}\,\mathrm{str}\left[k_1\; m^2\; \ln\left(\frac{m^2}{\mu^2}\right)\right] + \text{two loops.}$$

Pauli: one-loop finiteness (compensation):

Final stage:

$$\mathrm{str}(k_2') = \mathrm{str}(k_4') = 0.$$

$$k_{2/4}' = \text{“a right bloody mess”}.$$

$$K_{2/4} = (K_{2/4})_0 - \frac{1}{32\pi^2}\,\mathrm{str}\left[k_{2/4}'\;\ln\left(\frac{m^2}{\mu^2}\right)\right] + \text{two loops.}$$

Net result: You can keep all the one-loop effects of matter on the gravity sector finite, at the cost of strong constraints on the particle spectrum.

Maybe the price is too high?

- Pauli:
- “These requirements are so extensive that it is rather improbable that they are satisfied in reality.”

Sakharov:
one-loop dominance

Sakharov: one-loop dominance:

Sakharov's own interpretation was different.

- Set all tree-level constants to zero.
- One-loop physics is dominant.
- Newton constant induced at one-loop.
- Assume most dimensionless numbers of order one.
- Assume $\kappa \approx M_{\mathrm{Planck}}$;
  An explicit cutoff at the Planck scale.
- Assume $\ln(\kappa/\mu) \approx 137$ (?)

Sakharov: one-loop dominance:

Then:

$$\Lambda \approx -\frac{1}{64\pi^2}\,\mathrm{str}[I]\,\kappa^4; \qquad \mathrm{str}[I] \approx 0.$$

$$\frac{1}{G} \approx -\frac{1}{2\pi}\,\mathrm{str}[k_1]\,\kappa^2; \qquad \mathrm{str}[k_1] \approx -1.$$

$$K_2 \approx \frac{1}{32\pi^2}\,\mathrm{str}[k_2']\,\ln(\kappa^2/\mu^2) \approx 1.$$

$$K_4 \approx \frac{1}{32\pi^2}\,\mathrm{str}[k_4']\,\ln(\kappa^2/\mu^2) \approx 1.$$

Powers dominate over logarithms (wherever possible).
Coherent physical picture — but is there real predictive power?

Frolov–Furasev: one-loop calculability (more or less)

Frolov–Furasev (more or less):

Assume both Pauli compensation and Sakharov one-loop dominance.

$$\mathrm{str}(I) = \mathrm{str}(m^2) = \mathrm{str}(m^4) = 0.$$

$$\mathrm{str}(k_1) = \mathrm{str}(k_1\ m^2) = 0.$$

$$\mathrm{str}(k_2') = \mathrm{str}(k_4') = 0.$$

Then Λ and G are one-loop calculable:

$$\Lambda = +\frac{1}{64\pi^2}\,\mathrm{str}\left[m^4 \ln\left(\frac{m^2}{\mu^2}\right)\right] + \text{two loops.}$$

$$\frac{1}{G} = -\frac{1}{2\pi}\,\mathrm{str}\left[k_1\ m^2\ \ln\left(\frac{m^2}{\mu^2}\right)\right] + \text{two loops.}$$

$$K_{2/4} = -\frac{1}{32\pi^2}\,\mathrm{str}\left[k_{2/4}'\ \ln\left(\frac{m^2}{\mu^2}\right)\right] + \text{two loops.}$$

Just give me the particle spectrum...

Standard renormalization

Standard renormalization:

- Suppose you dislike Pauli compensation, Sakharov one-loop dominance, and Frolov–Furasev...
- Is there anything interesting you can say just using standard renormalization theory?
- (One-loop renormalizability in a curved space background; gravity itself is not quantized.)

Standard renormalization:

- Yes: renormalization allows you to absorb the one-loop divergences into the zero-loop bare quantities — but once and once only.
- You are not allowed to change your mind about how big an infinity you dump into the zero-loop bare quantities.
- That means that as you go through a phase transition or SSB, while the masses in the particle spectrum change, the coefficients of κ^4, κ^2, and $\log(\kappa^2/\mu^2)$ are not allowed to change.
- (Otherwise you are doing the equivalent of changing the zero of energy in the middle of the calculation.)

As you go thru a phase transition or SSB, some quantities invariant:

$$\delta \mathrm{str}(I) = \delta \mathrm{str}(m^2) = \delta \mathrm{str}(m^4) = 0.$$

$$\delta \mathrm{str}(k_1) = \delta \mathrm{str}(k_1\ m^2) = 0.$$

$$\delta \mathrm{str}(k_2') = \delta \mathrm{str}(k_4') = 0.$$

Other quantities have changes that are one-loop calculable:

$$\delta\Lambda = +\frac{1}{64\pi^2}\ \delta \mathrm{str}\left[m^4 \ln\left(\frac{m^2}{\mu^2}\right)\right] + \text{two loops.}$$

$$\delta\left(\frac{1}{G}\right) = -\frac{1}{2\pi}\ \delta \mathrm{str}\left[k_1\ m^2\ \ln\left(\frac{m^2}{\mu^2}\right)\right] + \text{two loops.}$$

$$\delta K_{2/4} \propto \delta \mathrm{str}\left[k_{2/4}'\ \ln\left(\frac{m^2}{\mu^2}\right)\right] + \text{two loops.}$$

Just give me the change in the particle spectrum...

Standard renormalization:

- These finiteness conditions are not satisfied by the SM.
- In the SM both Λ and G suffer infinite shifts going thru a phase transition or SSB.
- This is one version of the "cosmological constant problem", (also the "Newton constant problem"), necessitating BSM physics.
- Even if BSM physics patches up the finiteness conditions, you still expect finite shifts in Λ and G.
- Can this be made compatible with experiment/observation?

Standard renormalization: Running couplings...

Provided masses (and k_i) are not changing:

$$\Lambda(\mu) = \Lambda(\mu_0) - \frac{1}{64\pi^2} \operatorname{str}\left[m^4\right] \ln\left(\frac{\mu^2}{\mu_0^2}\right) + \text{two loops.}$$

$$\frac{1}{G(\mu)} = \frac{1}{G(\mu_0)} + \frac{1}{2\pi} \operatorname{str}\left[k_1\; m^2\right]\; \ln\left(\frac{\mu^2}{\mu_0^2}\right) + \text{two loops.}$$

$$K_{2/4}(\mu) = K_{2/4}(\mu_0) + \frac{1}{32\pi^2}\operatorname{str}\left[k'_{2/4}\right]\; \ln\left(\frac{\mu^2}{\mu_0^2}\right) + \text{two loops.}$$

Standard logarithmic running...
(And Pauli compensation would imply no one-loop running...)

Standard renormalization: β functions...

Definitions:

$$\beta_g \equiv \mu \frac{\partial g}{\partial \mu}; \qquad \gamma_m \equiv \frac{\mu}{m}\frac{\partial m}{\partial \mu}.$$

To maintain one-loop renormalizability in a curved space background:

$$\mathrm{str}\left[m^2\, \gamma_m\right] = 0 = \mathrm{str}\left[m^4\, \gamma_m\right].$$

$$\mathrm{str}\left[\beta_{k_1}\right] = 0 = \mathrm{str}\left[\gamma_{(k_1 m^2)}\right].$$

$$\mathrm{str}\left[\beta_{k_2'}\right] = 0 = \mathrm{str}\left[\beta_{k_4'}\right].$$

These are the differential versions of the invariance constraints.

Standard renormalization: β functions...

The β functions are:

$$\beta_\Lambda = -\frac{1}{32\pi^2}\,\mathrm{str}\left[m^4\right] + \text{“}\gamma_m\text{”} + \text{two loops.}$$

$$\beta_{\frac{1}{G}} = +\frac{1}{\pi}\,\mathrm{str}\left[k_1\ m^2\right] + \text{“}\gamma_m\text{”} + \text{“}\gamma_{(k_1 m^2)}\text{”} + \text{two loops.}$$

$$\beta_{K_{2/4}} = +\frac{1}{16^2\pi}\mathrm{str}\left[k'_{2/4}\right] + \text{“}\gamma_m\text{”} + \text{“}\beta_{k'_{2/4}}\text{”} + \text{two loops.}$$

There is a whole lot of unexplored territory here...

The story so far

The story so far:

One-loop matter leads to shifts in Λ and G.
Depending on your choices you can adopt:

- Sakharov: one-loop dominance.
- Pauli (weak): one-loop finiteness.
- Pauli (strong): all-loop finiteness.
- Frolov–Furasev: one-loop calculable.
- "Standard": one-loop calculable changes.

Even if Einstein gravity is not there at zero loops, it will automatically be generated at one-loop.

Can we now graft these ideas onto either:

(1) Lorentzian lattice quantum gravity, or

(2) the Lorentzian manifolds arising in "analogue gravity"?

Lorentzian Lattice Quantum Gravity

Based on work by Jan Ambjorn/ Renate Loll and collaborators.
"Large" and "smooth" lattices seem generic.
Quantize ordinary matter on these large smooth manifolds.
Λ: Pauli —

$$\mathrm{str}(I) = \mathrm{str}(m^2) = \mathrm{str}(m^4) = 0.$$

$$\Lambda = \Lambda_0 + \frac{1}{64\pi^2}\,\mathrm{str}\left[m^4 \ln\left(\frac{m^2}{\mu^2}\right)\right] + \text{two loops.}$$

G: Sakharov —

$$\frac{1}{G} \approx -\frac{1}{2\pi}\,\mathrm{str}[k_1]\,\kappa^2; \qquad \mathrm{str}[k_1] \approx -1.$$

Once you have "large" "smooth" manifolds, GR seems automatic?

Examples

Lorentzian geometry for free:

Single scalar field.

Lagrangian:

$$\mathcal{L}(\partial_\mu \phi, \phi).$$

Action:

$$S[\phi] = \int \mathrm{d}^{d+1}x \; \mathcal{L}(\partial_\mu \phi, \phi).$$

Euler–Lagrange equations:

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0.$$

Linearize the field around a solution:

$$\phi(t, \vec{x}) = \phi_0(t, \vec{x}) + \epsilon \phi_1(t, \vec{x}) + \frac{\epsilon^2}{2} \phi_2(t, \vec{x}) + O(\epsilon^3).$$

Example 1:

Linearized action

$$
\begin{aligned}
S[\phi] &= S[\phi_0] \\
&+ \frac{\epsilon^2}{2} \int \mathrm{d}^{d+1}x \left[\left\{ \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi)\, \partial(\partial_\nu \phi)} \right\} \partial_\mu \phi_1 \; \partial_\nu \phi_1 \right. \\
&\qquad \left. + \left(\frac{\partial^2 \mathcal{L}}{\partial \phi \; \partial \phi} - \partial_\mu \left\{ \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi)\; \partial \phi} \right\} \right) \phi_1 \; \phi_1 \right] \\
&+ O(\epsilon^3).
\end{aligned}
$$

Linear pieces $[O(\epsilon)]$ vanish by equations of motion.

Quadratic in $\phi_1 \Rightarrow$ field-theory normal modes.

Example 1:

Linearized equations of motion:

$$\partial_\mu \left(\left\{ \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi)\, \partial(\partial_\nu \phi)} \right\} \partial_\nu \phi_1 \right) - \left(\frac{\partial^2 \mathcal{L}}{\partial \phi\, \partial \phi} - \partial_\mu \left\{ \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi)\, \partial \phi} \right\} \right) \phi_1 = 0.$$

Formally self-adjoint.

Example 1:

Geometrical interpretation:

$$[\Delta(g(\phi_0)) - V(\phi_0)]\,\phi_1 = 0.$$

Metric:

$$\sqrt{-g}\; g^{\mu\nu} \equiv \left\{ \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi)\;\partial(\partial_\nu \phi)} \right\}\Bigg|_{\phi_0}.$$

Potential:

$$V(\phi_0) \;=\; \frac{1}{\sqrt{-g}} \left(\frac{\partial^2 \mathcal{L}}{\partial\phi\;\partial\phi} - \partial_\mu \left\{ \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu\phi)\;\partial\phi} \right\} \right).$$

And

linearization $\Rightarrow$ metric;

hyperbolic $\Rightarrow$ pseudo-Riemannian;

parabolic $\Rightarrow$ degenerate;

elliptic $\Rightarrow$ Riemannian.

Example 2:

Barotropic irrotational inviscid fluid dynamics:

Lagrangian (two fields):

$$\mathcal{L} = -\rho \, \partial_t \theta - \frac{1}{2} \rho (\nabla \theta)^2 - \int_0^\rho d\rho' \; h(\rho').$$

$$h(\rho) = h[p(\rho)] = \int_0^{p(\rho)} \frac{dp'}{\rho(p')}.$$

Vary $\rho \Rightarrow$ Bernoulli equation (Euler equation).

$$\partial_t \theta + \frac{1}{2} (\nabla \theta)^2 + h(\rho) = 0.$$

Vary $\theta \Rightarrow$ continuity equation.

$$\partial_t \rho + \nabla (\rho \, \nabla \theta) = 0.$$

Use the Bernoulli equation to algebraically eliminate ρ:

$$\rho = h^{-1}(z) = h^{-1}\left(-\partial_t\theta - \frac{1}{2}(\nabla\theta)^2\right).$$

Reduced Lagrangian:

$$\mathcal{L}(z) = z\;\rho(z) - \int_0^{\rho(z)} d\rho'\; h(\rho').$$

But:

$$p(\rho) = \rho\; h(\rho) - \int_0^{\rho} d\rho'\; h(\rho').$$

Example 2:

Finally:

$$\mathcal{L} = p(\rho(z)) = p\left(h^{-1}\left(-\partial_t\theta - \frac{1}{2}(\nabla\theta)^2\right)\right)$$
$$= \mathcal{L}\left(-\partial_t\theta - \frac{1}{2}(\nabla\theta)^2\right).$$

This reduces the Lagrangian to the form of Example 1.
Therefore there is a metric hiding here just waiting to be found...

Apply the result of Example 1:

$$\sqrt{-g}\; g^{\mu\nu} \equiv f^{\mu\nu} \equiv \left\{ \frac{\partial^2 \mathcal{L}}{\partial(\partial_\mu \phi)\; \partial(\partial_\nu \phi)} \right\}\Bigg|_{\phi_0}.$$

Therefore:

$$f^{\mu\nu} = -\rho\; c_s^{-2} \left[\begin{array}{ccc} -1 & \vdots & -\nabla^i\theta \\ \cdots\cdots & \cdot & \cdots\cdots\cdots\cdots\cdots\cdots \\ -\nabla^j\theta & \vdots & c_s^2\; \delta^{ij} - \nabla^i\theta\; \nabla^j\theta \end{array} \right].$$

This is equivalent to the standard (d+1) dimensional "acoustic metric". Use

$$g^{\mu\nu} = |\mathrm{det} f|^{-1/(d-1)}\; f^{\mu\nu}.$$

Example 2:

When the dust settles

$$g^{\mu\nu} \propto \left[\begin{array}{c|c} -1 & -\nabla^i\theta \\ \hline -\nabla^j\theta & c_s^2\,\delta^{ij} - \nabla^i\theta\,\nabla^j\theta \end{array} \right].$$

Inverting

$$g_{\mu\nu} \propto \left[\begin{array}{c|c} -(c_s^2 - [\nabla\theta]^2) & -\nabla_i\theta \\ \hline -\nabla_j\theta & \delta_{ij} \end{array} \right].$$

Equivalently

$$\mathrm{d}s^2 \propto -c_s^2\,\mathrm{d}t^2 + (\mathrm{d}x - \nabla\theta\,\mathrm{d}t)^2.$$

Natural way of assigning a pseudo-Riemannian (Lorentzian) metric to this physical system.

This metric governs the propagation of linearized fluctuations — in this context, sound waves.

Summary so far

Lorentzian geometries are "natural".

Lorentzian geometries show up in many a priori unexpected places.
The reason the previous two examples are interesting is because they are part of a much more general pattern.

With many interacting fields there are severe technical complications...
Courant and Hilbert:

> *It may be remarked that the present state of the theory of algebraic surfaces does not permit entirely satisfactory applications to the questions of reality of geometric structures which confront us here.*

Not quite the Einstein equations

Not quite the Einstein equations:

- So analogue methods (linearization, normal modes) naturally lead to Lorentzian manifolds.
- Sakharov's ideas (quantized free fields) naturally lead to an Einstein–Hilbert term.
- Put these ideas together:
- Is Einstein gravity an automatic consequence of QFT?
- Not so fast...
- The fly in the ointment is this:
- How general a class of Lorentzian geometries emerges form the analogue methods?

Not quite the Einstein equations:

If the metric depends implicitly on a set of fields ϕ, then the EOM are obtained by the chain rule:

$$\frac{\delta \mathcal{S}(g(\phi),\phi)}{\delta g_{ab}(\phi)} \; \frac{\delta g_{ab}}{\delta \phi} + \frac{\delta \mathcal{S}(g(\phi),\phi)}{\delta \phi} = 0.$$

This implies:

$$\left\{ \kappa \; G^{ab} + \Lambda \; g^{ab} - 8\pi \; T^{ab} \right\} \; \frac{\delta g_{ab}}{\delta \phi} = \text{tree} + \text{two-loop} + \text{other}.$$

If you adopt one-loop dominance:

$$\left\{ \kappa \; G^{ab} + \Lambda \; g^{ab} - 8\pi \; T^{ab} \right\} \; \frac{\delta g_{ab}}{\delta \phi} \approx 0.$$

But this is still not the full Einstein equations.
It's the Regge–Teitelboim–Deser submanifold restriction.
(An early "braneworld" model.)

Conclusions

Conclusions:

Basic message:

- Hyperbolic PDE $\Rightarrow$ Lorentzian geometry.
- "Effective metrics" hiding in the woodwork.
- Sakharov $\Rightarrow$ Einstein–Hilbert term.
- Not quite the Einstein equations?
- Still, it's suggestive and promising....

End of Lecture 4.