QFT School: Lecture 1

0. Motivation

- condensed-matter-physics perspective on QFT: what is the conceptual relationship?

high-energy ← $\boxed{\text{QFT}}$ → condensed-matter

particle physics		quasi-particle physics	
electrons & photons	QED	phonons	lattice vibrations
		plasmons	charge-density oscillations
quarks	QCD	fractionally charged / spinless / bound "electrons"	Laughlin-FQHE, strongly correlated itinerant magnets, Cooper pairs
Higgs			
axions			
⋮		⋮	

$\boxed{\text{reductionism}} \longleftrightarrow \boxed{\text{emergence}}$ Venema et al., Nat. Phys. 12, 1085 (16)

- universality: physics = knowledge about concepts, not things!

massless particle $\xleftrightarrow{\text{in vacuum}}$ photon $\xleftrightarrow{\text{in waveguide}}$ massive particle

massless particle $\xleftrightarrow{\text{in graphene}}$ electron $\xleftrightarrow{\text{in vacuum}}$ massive particle

transferable models / mathematical methods / analogies

- mutual inspiration: Anderson-Higgs mechanism
  renormalisation group
  topological effects

- Lit: A Altland and B D Simons, Condensed Matter Field Theory (Cambridge, 2nd Ed. 2010)

②

1. Euclidean path-integral formalism

- recap: real-time Feynman path integral for the quantum-mechanical propagator

$$|\Psi(t'')\rangle = \hat{U}(t'',t')\,|\Psi(t')\rangle \qquad \hat{U}(t'',t') = e^{-\frac{i}{\hbar}\hat{H}(t''-t')}\,\Theta(t''-t')$$

(time-evolution operator)

$$\Psi(q'',t'') \equiv \langle q''|\Psi(t'')\rangle = \langle q''|\hat{U}(t'',t')\underset{\mathbb{1}=\int dq'\,|q'\rangle\langle q'|}{\wedge}|\Psi(t')\rangle$$

$$= \int dq'\,\underline{\underline{\langle q''|\hat{U}(t'',t')|q'\rangle}}\,\Psi(q',t')$$

"propagator" relates initial and final quantum amplitudes

$\Rightarrow$ step 1: $\hat{U}(t'',t') \equiv e^{-\frac{i}{\hbar}\hat{H}(t''-t')} = \left(e^{-\frac{i}{\hbar}\hat{H}\delta t}\right)^{N+1} \qquad \delta t = \frac{t''-t'}{N+1}$

step 2: $\langle q''|\hat{U}(t'',t')|q'\rangle = \int \prod_{j=1}^{N} dq_j\;\langle q''|e^{-\frac{i}{\hbar}\hat{H}\delta t}|q_N\rangle \cdots$
$\cdots \langle q_1|e^{-\frac{i}{\hbar}\hat{H}\delta t}|q'\rangle$

step 3: $\langle q_{j+1}|e^{-\frac{i}{\hbar}\hat{H}\delta t}|q_j\rangle \simeq e^{\frac{i}{\hbar}\delta t\,L(q_j,\dot{q}_j)}$

$\Rightarrow$ final result:

$$\langle q''|\,e^{-\frac{i}{\hbar}\hat{H}(t''-t')}|q'\rangle = \int_{\substack{q(t'')=q''\\ q(t')=q'}} \mathcal{D}q\; e^{\frac{i}{\hbar}\int_{t'}^{t''}dt\,L(q,\dot{q})}$$

- quantum statistical mechanics: partition function Z

$$Z = \mathrm{Tr}\left(e^{-\beta\hat{H}}\right) \equiv \int dq\,\langle q|e^{-\beta\hat{H}}|q\rangle$$

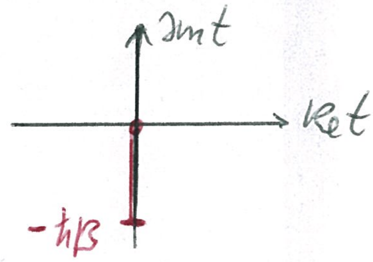
$\Rightarrow$ yields thdy potentials: $F = -\frac{1}{\beta}\ln Z \quad ; \qquad \beta = \frac{1}{k_B T}$

$\Rightarrow$ notice analogy: $\langle q | e^{-\beta \hat{H}} | q \rangle \equiv \underline{\underline{\langle q | e^{-\frac{i}{\hbar}\hat{H}t} | q \rangle}}$

propagator for imaginary time
$t = -i\hbar\beta$; $q'' = q' = q$

thus expect path-integral representation for partition function:

$$Z = \int\limits_{q(-i\hbar\beta)=q(0)} \mathcal{D}q \; e^{\frac{i}{\hbar}\int_0^{-i\hbar\beta} dt\, L(q,\dot{q})}$$



for convenience, define $\tau = it$ ... "imaginary time" and find

$$Z = \int\limits_{q(\hbar\beta)=q(0)} \mathcal{D}q \; e^{-\frac{1}{\hbar}\int_0^{\hbar\beta} d\tau\, L_E(q,\dot{q})}$$

Euclidean path integral

with the Euclidean Lagrangian $L_E(q,\dot{q}) = -L(q,\dot{q})\big|_{t\to -i\tau}$

example: see ③a

$\dot{q}$: $dq/d\tau$; $\dot{q}$: dq/dt

- application: path integral for spin S

$\Rightarrow$ Hamiltonian for spin: $\hat{\mathcal{H}} = \vec{B}\cdot\hat{\vec{S}}$

magnetic field

vector of spin-S operators $(\hat{S}_1, \hat{S}_2, \hat{S}_3)$

$\Rightarrow$ partition function: $Z = \mathrm{Tr}\left(e^{-\beta\hat{\mathcal{H}}}\right)$

$\Rightarrow$ construction of (imaginary-time) path integral:

- step 1: $e^{-\beta\hat{\mathcal{H}}} \equiv \left(e^{-\delta\tau\hat{\mathcal{H}}}\right)^{N+1}$

3a

- example : particle in potential $V(q)$

$$L(q,\dot{q}) = \frac{m}{2}\dot{q}^2 - V(q)$$

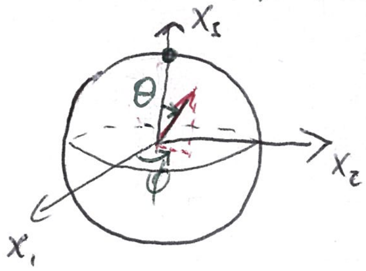
$$L_E(q,\dot{q}) = -\frac{m}{2}\left(\frac{\partial}{\partial(-i\tau)}q\right)^2 + V(q) \equiv \frac{m}{2}\dot{q}^2 + V(q)$$

"Euclidean Lagrangian = Minkowski Hamiltonian written as function of $\dot{q}$, q ..."

(4)

- for step 2: need resolution of identity in terms of generalised coordinates: $\mathbb{1} = \int dq\, |q\rangle\langle q|$

use representation of most general spin state in terms of Euler angles:



$$|\phi,\theta\rangle \equiv e^{-i\phi \hat{S}_3}\, e^{-i\theta \hat{S}_2}\, |\uparrow\rangle$$

maximum-eigenvalue $\hat{S}_3$-projection state

thus have
$$Z = \int \prod_{j=1}^{N} d\phi_j\, d\theta_j\, \langle \phi_1,\theta_1| e^{-\delta\tau \hat{\mathcal{H}}} |\phi_N,\theta_N\rangle \dots$$
$$\dots \langle \phi_2,\theta_2| e^{-\delta\tau \hat{\mathcal{H}}} |\phi_1,\theta_1\rangle$$

- step 3: $\langle \phi_{j+1},\theta_{j+1}| e^{-\delta\tau \hat{\mathcal{H}}} |\phi_j,\theta_j\rangle$ $\qquad e^{-\delta\tau \mathcal{H}} \simeq 1 - \delta\tau \mathcal{H}$

$$\simeq \langle \phi_{j+1},\theta_{j+1}|\phi_j,\theta_j\rangle - \delta\tau \langle \phi_{j+1},\theta_{j+1}|\hat{\mathcal{H}}|\phi_j,\theta_j\rangle$$
$$+ 1 - \underbrace{\langle \phi_j,\theta_j|\phi_j,\theta_j\rangle}_{\equiv 1 \text{ by def.}}$$

$$\simeq \exp\left\{ \delta\tau \cdot \left(\frac{\langle \phi_{j+1},\theta_{j+1}| - \langle \phi_j,\theta_j|}{\delta\tau} |\phi_j,\theta_j\rangle - \langle \phi_{j+1},\theta_{j+1}|\hat{\mathcal{H}}|\phi_j,\theta_j\rangle \right) \right\}$$

$\Rightarrow$ continuum limit ($\delta\tau \to 0$, $N \to \infty$) yields final result

$$Z = \int \mathcal{D}\phi\, \mathcal{D}\theta\; e^{-\int_0^\beta d\tau\, L_E(\phi,\dot{\phi},\theta,\dot{\theta})}$$

with $L_E = -(\partial_\tau \langle \phi,\theta|)|\phi,\theta\rangle + \langle \phi,\theta| \vec{B}\cdot\vec{S} |\phi,\theta\rangle$

⑤

more explicitly, using the explicit form of the states $|\phi,\theta\rangle$, we find

$$(\partial_\tau\langle\phi,\theta|)|\phi,\theta\rangle \equiv i\langle\uparrow|\left(\dot\theta\hat S_2 e^{i\theta\hat S_2}e^{i\phi\hat S_3}+\dot\phi e^{i\theta\hat S_2}\hat S_3 e^{i\phi\hat S_3}\right) e^{-i\phi\hat S_3}e^{-i\theta\hat S_2}|\uparrow\rangle$$

spin identity: $\boxed{e^{-i\varphi\hat S_i}\hat S_j e^{i\varphi\hat S_i} = \hat S_j\cos\varphi+\varepsilon_{ijk}\hat S_k\sin\varphi}$

$$= i\underbrace{\langle\uparrow|\hat S_2|\uparrow\rangle}_{\equiv 0}\dot\theta + i\underbrace{\langle\uparrow|\underbrace{e^{i\theta\hat S_2}\hat S_3 e^{-i\theta\hat S_2}}_{\equiv \hat S_3\cos\theta-\hat S_1\sin\theta}|\uparrow\rangle}_{\equiv S\cos\theta}\dot\phi$$

$$= iS\dot\phi\cos\theta$$

and

$$\langle\phi,\theta|\vec B\cdot\vec S|\phi,\theta\rangle \equiv \vec B\cdot\underbrace{\langle\phi,\theta|\vec S|\phi,\theta\rangle}_{\equiv\vec n(\phi,\theta)=S(\sin\theta\cos\phi,\,\sin\theta\sin\phi,\,\cos\theta)}$$

yielding finally

$$L_E = S\left(i[1-\cos\theta]\dot\phi + \vec B\cdot\vec n(\phi,\theta)\right)$$

↑ introduced for consistency with other derivations; is full time derivative and, hence, zero contribution to action!

$\Rightarrow$ Euler-Lagrange eqs:
$$\frac{d}{d\tau}\frac{\partial L_E}{\partial\dot\phi}-\frac{\partial L_E}{\partial\phi}=0$$
$$\frac{d}{d\tau}\frac{\partial L_E}{\partial\dot\theta}-\frac{\partial L_E}{\partial\theta}=0$$

turn out to be equivalent to
$$i\partial_\tau\vec n = \vec B\times\vec n$$

embodies spin precession!