QFT School: Lecture 2

2. Topological field theories: Chern-Simons terms

- MG previously discussed examples for topological field theories: Wess-Zumino terms, related to Θ-terms that describe winding numbers / topological sectors!

- here we will consider topological effects in a gauge field theory: relevant for understanding fractional quantum Hall effect and topological / Chern insulators!

2.1 Reminder: Coupling matter to electromagnetism

- charged particle subject to electromagnetic fields:

Hamiltonian $\mathcal{H}(\vec{r},\vec{p}) = \dfrac{[\vec{p} - eA(\vec{r})]^2}{2m} + eV(\vec{r})$

$\vec{A}(\vec{r})$: vector potential

$V(\vec{r})$: electrostatic potential

$$\Rightarrow \frac{\partial \mathcal{H}}{\partial \vec{p}} = \dot{\vec{r}} \equiv \frac{\vec{p} - e\vec{A}(\vec{r})}{m}$$

$$\Rightarrow \text{Lagrangian } \mathcal{L} = \vec{p}\cdot\dot{\vec{r}} - \mathcal{H}(\vec{r},\vec{p}) \equiv (\overbrace{\vec{p} - e\vec{A}}^{m\dot{\vec{r}}} + e\vec{A})\cdot\dot{\vec{r}} - \frac{m}{2}\dot{\vec{r}}^2 - eV$$

$$= \frac{m}{2}\dot{\vec{r}}^2 + \underbrace{e\dot{\vec{r}}\cdot\vec{A} - eV}$$

$$\equiv \int d^d r' \left[\underbrace{e\,\delta(\vec{r}' - \vec{r})\dot{\vec{r}}}_{\vec{j}(\vec{r}')\ldots\text{ current density}} \cdot \vec{A}(\vec{r}') - \underbrace{e\,\delta(\vec{r}' - \vec{r})}_{\rho(\vec{r}')\ldots\text{ charge density}} V(\vec{r}') \right]$$

- term $\int d^{d+1}x \; j^\mu A_\mu$ generally referred to as "minimal coupling"

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- dynamics of vector potential: Maxwell term

$$\mathcal{L}_M = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \quad \text{with } F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

field tensor

$\Rightarrow$ $\mathcal{L}_M + j_\mu A^\mu$ yields Maxwell eqs $\partial_\mu F^{\mu\nu} = J^\nu$

Maxwell electromagnetism is possible in any dimension d

eg, d=2: $$F_{\mu\nu} = \begin{pmatrix} 0 & -\mathcal{E}_1 & -\mathcal{E}_2 \\ \mathcal{E}_1 & 0 & B \\ \mathcal{E}_2 & -B & 0 \end{pmatrix}$$

2.2 Specialty of even d: Possibility of Chern-Simons term

- consider d=2: as an alternative to Maxwell-type Lagrangian, can have a different combination of vector-potential components:

$$\mathcal{L}_{CS} = \frac{\varkappa}{2} \varepsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho$$

Lorentz-invariant ✓
gauge-invariant ✓
local ✓

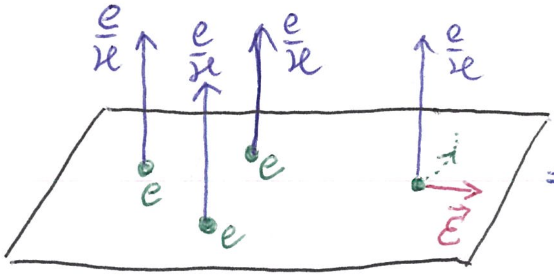
- classical Euler-Lagrange eqs.: $\frac{\varkappa}{2} \varepsilon^{\mu\nu\rho} F_{\nu\rho} = j^\mu$

more explicitly:

$$\rho = \varkappa B \quad \longleftarrow \text{magnetic flux bound to matter}$$

$$j^i = \varkappa \varepsilon^{ij} \mathcal{E}_j \quad \longleftarrow \text{quantised Hall effect}$$

($\varkappa$... Hall conductance)



$\Rightarrow$ flux tubes of strength $\frac{e}{\varkappa}$ attached to point particles

$\Rightarrow \vec{J} \perp \vec{\mathcal{E}}$ like in Hall effect

- CS term induces no dynamics, just introduces constraint! topological

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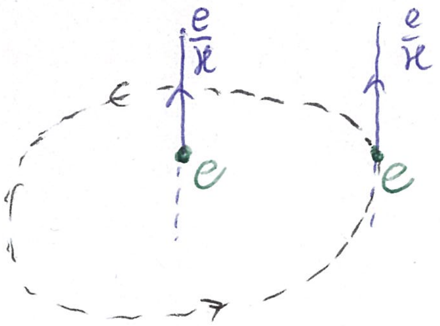
• ramification of flux-tube attachment: realisation of anyons



$$\Psi(1,2) \xrightarrow[\text{encircles particle 2}]{\text{particle 1}} \Psi(1,2)\cdot e^{i\frac{e^2}{\kappa}}$$

$$\Psi(1,2) \xrightarrow[\text{process}]{\text{anyon}} \Psi(2,1) \equiv \Psi(1,2)\cdot e^{i\frac{e^2}{2\kappa}}$$

$\Rightarrow$ in 3D: $\Psi(2,1) = \pm\,\Psi(1,2)$ bosons or fermions

in 2D: $\Psi(2,1) = e^{in\theta}\,\Psi(1,2)$ arbitrary θ

n... # of windings for exchange-path scenario

$\Rightarrow$ particle coupled to CS field: $\theta \equiv \frac{e^2}{2\kappa}$

• sketch of proof: see Altland and Simons, Sec 9.5.2

$$Z = \int \mathcal{D}\Psi\, \mathcal{D}A^\mu\, e^{-S_0[\Psi] + \int d^3x\, j_\mu A^\mu - S_{CS}[A^\mu]}$$

$$\equiv \int \mathcal{D}\tilde{\Psi}\; e^{-S_0[\Psi] + \frac{1}{2}\int d^3x\, d^3x'\, j_\mu(x)\, j_\nu(x')\, \Delta^{\mu\nu}_{CS}}$$

$$\Delta^{\mu\nu}_{CS} = \langle A^\mu(x)\, A^\nu(x)\rangle_{CS}$$

$$\equiv \sum_{n\in\mathbb{Z}} \int \mathcal{D}\tilde{\Psi}_n\; e^{-S_0[\Psi]}\; e^{i\frac{e^2}{\kappa}n}$$

"intertwinedness of particle trajectories"

topological sectors generated by CS term !!

• application: Fractional Quantum Hall effect $\kappa = \frac{e^2}{4\pi\hbar}$

flux attachment used to relate FQHE of electrons to Integer Quantum Hall Effect of composite fermions