(Some) Lowness notions in the c.e. sets

Peter Cholak

January, 2017

Joint with Rachel Epstein
Computability and Complexity Symposium 2017
New Zealand
In honor of Rod Downey's 60th birthday

A beginning

Let A be a coinfinite c.e. set. In the late 70's, Soare showed that if A is low then the collection of all c.e. sets which contain A under inclusion (the outside of A) are isomorphic to the collection of all c.e. sets under inclusion (everything).

Definition

The outside of A is denoted $\mathcal{L}(A)$ which is the structure $\{W_e \cup A | e \in \omega\}$ under inclusion. $\mathcal{E}$ is the structure $\{W_e | e \in \omega\}$ under inclusion.

Note that if $A = \emptyset$ then $\mathcal{L}(A) = \mathcal{E}$.

Soare's Result

Theorem (Soare)

If A is low then $\mathcal{L}(A)$ and $\mathcal{E}$ are isomorphic.

Question

For which A are $\mathcal{L}(A)$ and $\mathcal{E}$ are isomorphic?

We will see that this question is about lowness notions. If A realizes one of our lowness notions then we want that $\mathcal{L}(A)$ and $\mathcal{E}$ are isomorphic. If possible even more.

What would an isomorphism Φ look like?

We will think of $\mathcal{L}(A)$ as living in ω and $\mathcal{E}$ as living in another copy of ω, denoted $\tilde{\omega}$. Denote the elements, $W_e \cup A$ of $\mathcal{L}(A)$ as U_e and similarly the elements of $\mathcal{E}$ as V_e. U_e and V_e will be called RED or given sets. So $\Phi(U_e)$ must be a c.e. set living in $\tilde{\omega}$, let's denote it as $\widehat{U}_e$. Similarly $\Phi^{-1}(V_e) = \widehat{V}_e$. $\widehat{U}_e$ and $\widehat{V}_e$ will be called BLUE or built sets.

Φ is an isomorphism iff the venn diagrams of the sets living in ω and $\tilde{\omega}$ correspond. Hence, for all e, the cardinality of each region (in the diagram) formed by the sets U_i and $\widehat{V}_i$, for $i \leq e$, have the same cardinality of the corresponding region formed by the sets $\widehat{U}_i$ and V_i, for $i \leq e$.

Soare showed that it is enough that a region in ω and it's corresponding region $\tilde{\omega}$ either both be infinite or both be finite.

How was lowness used? Part I: Informational

In short, by helping build the BLUE sets. We will construct the BLUE sets inductively. The regions are differences of c.e. sets and we need to know if they are infinite. We break this into two questions using the fact that all the c.e. sets have an uniform computable enumeration.

Do infinitely many balls (elements of $\overline{A}$ or $\tilde{\omega}$) enter into a region? The set of balls which enter a region is c.e.. Given x, is there a stage s where x is in that region (in some c.e. sets and out of others). This first question corresponds to asking if the above c.e. set is infinite outside A or in $\tilde{\omega}$. A Π^0_2 question in $\tilde{\omega}$ and a Π^A_2 question in ω. Since A is low this Π^A_2 question is Δ^0_3.

Do almost all these balls later leave this region? Every $x > k$ in the region at some stage later enters some new c.e. set not in the region. The second question is Σ^0_3.

Semilow$_2$

Definition

B is *semilow$_2$* iff $\{e | W_e \cap B \text{ is infinite}\} \leq_T \mathbf{0}''$.

For the first question to be Δ^0_3 it is enough for $\overline{A}$ to be semilow$_2$. If A is low$_2$ then $\overline{A}$ is semilow$_2$.

Lemma

Every nonlow$_2$ c.e. degrees contains c.e. sets A and B such that $\overline{A}$ is semilow$_2$ and $\overline{B}$ is not semilow$_2$.

This type of lemma is true for all reasonable boolean combinations of the lowness notions we consider.

Lemma (Follows from another Soare)

There are sets A where $\overline{A}$ is semilow$_2$ but $\mathcal{L}(A)$ is not isomorphic to $\mathcal{E}$.

So all the lowness properties of A that we are interested in will imply that $\overline{A}$ is semilow$_2$ but they must be stronger.

Why not low$_3$?

First asking whether $W_e \cap \overline{A}$ (or W_e^A) is infinite need not be Δ^0_3 if A is not low$_2$.

Theorem (Shoenfield)

Every c.e. nonlow$_2$ degree contains a c.e. set A such that $\mathcal{L}(A)$ and $\mathcal{E}$ are not isomorphic.

How was lowness used? Part II: The outside of A

Assume that V_0 is infinite and coinfinite. How can we ensure that $\Phi^{-1}(V_0) = \widehat{V}_0$ is infinite and coinfinite outside A? How do we build $\widehat{V}_0$ so that $\widehat{V}_0 \cap \overline{A}$ is infinite and coinfinite? We cannot enumerate everything into $\widehat{V}_0$. What stops all the elements enumerated into $\widehat{V}_0$ from entering A? How do we get *access* to elements of $\overline{A}$?

Definition

$\overline{A}$ is *semilow* iff $\{e | W_e \cap \overline{A} \neq \emptyset\} \leq_T \mathbf{0}'$.

This a $\Sigma_1^{\overline{A}}$ question. If A is low then this question is Δ_2^0.

Use semilowness of $\overline{A}$ and the limit lemma to uniformly split ω into the disjoint union of *finite* sets F_i such that, for all i, $F_i \cap \overline{A}$ is nonempty. At stage s if our approximation of $\mathbf{0}'$ says that the set $(\omega - \bigsqcup_{i<e} F_e) \cap \overline{A}$ is nonempty but $F_e \cap \overline{A}$ is empty, put the element x of ω which enters at stage s into F_e (for the least such e), otherwise x goes into F_s. Let $\widehat{V}_0$ be the union of F_{2i}, for all i. F_i provide access.

Semilow

Theorem (Soare)

If $\overline{A}$ is semilow then $\mathcal{L}(A)$ and $\mathcal{E}$ are (effectively) isomorphic.

Theorem (Mostly Russell Miller and Epstein)

If $\overline{A}$ is semilow then A is in the same orbit as a low set.

Consider Φ as an isomorphism between $\mathcal{L}(\emptyset)$ and $\mathcal{E}$ where $\Phi(A)$ is low. Here we say $A \subset \omega$ and $\Phi(A) = \hat{A} \subset \tilde{\omega}$ are automorphic. The *issue* is *covering* the regions that are infinite and fall into A. A key aspect of being able to cover is that the F_i on the previous slide are finite.

Miller showed that $\hat{A}$ can avoid the cone above a noncomputable c.e. set C.

Theorem (Epstein)

There is a properly low$_2$ degree such that every c.e. set in that degree is automorphic to a low set.

Characterization?

Question (Soare)

What c.e. sets are (effectively) automorphic to a low set?

Just the c.e. sets whose complements are semilow? No.

Theorem

There is A such that A is effectively automorphic to a low set but $\overline{A}$ is not semilow.

Semilow$_{1.5}$

Definition (Maass)

B is *semilow*$_{1.5}$ iff
$\{e|W_e \cap B \text{ is infinite}\} \leq_m \{e|W_e \text{ is infinite}\} = INF$.

Stronger than semilow$_2$, weaker than semilow.

Theorem (Maass)

If $\overline{A}$ is semilow$_{1.5}$ then $\mathcal{L}(A)$ and $\mathcal{E}$ are isomorphic.

Semilow$_{1.5}$ and the outside of A

Definition (Maass)

A has the *outer splitting property* iff there are two computable functions f and g such that, for all e,

1. $W_e = W_{f(e)} \sqcup W_{g(e)}$,
2. $W_{f(e)} \cap \overline{A}$ is finite, and
3. if $W_e \cap \overline{A}$ is infinite then $W_{f(e)} \cap \overline{A}$ is nonempty.

Lemma (Maass)

If $\overline{A}$ is semilow$_{1.5}$ then A has the outer splitting property.

Proof.

Using that they both are m-reducible to INF, play $W_{f(e)} \cap \overline{A}$ is empty against $W_{f(e)} \cap \overline{A}$ is infinite. □

The outer splitting property gives access to elements of $\overline{A}$ via $W_{f(e)}$. But $W_{f(e)}$ need not be finite. If $W_{f(e)}$ is guaranteed to always be finite then $\overline{A}$ is semilow.

Orbits, Low, and Low$_2$

Theorem (Harrington and Soare)

There is a nonempty property $NL(A)$ definable in the structure $\mathcal{E}$ such that if $NL(A)$ holds then A does not have a semilow complement. There is a set A with semilow$_{1.5}$ complement and $NL(A)$. So A is not automorphic to a low set.

Question

Are c.e. sets with semilow$_{1.5}$ complement automorphic to a low$_2$ set?

Question

If A has semilow$_{1.5}$ complement and $NL(A)$ holds then is A automorphic to a low$_2$ set?

Question

Are there c.e. sets A such that A has semilow$_{1.5}$ complement, $NL(A)$ holds, and A is nonlow$_2$?

Beyond Semilow$_{1.5}$

Theorem (Cholak)

If A has the outer splitting property and $\overline{A}$ is semilow$_2$ then $\mathcal{L}(A)$ and $\mathcal{E}$ are isomorphic.

Question

If A is low$_2$ then are $\mathcal{L}(A)$ and $\mathcal{E}$ isomorphic?

Lemma (Downey, Jockusch, and Schupp, 2013)

Every nonlow c.e. degree contains a c.e. set without the outer splitting property.

So the above theorem does not answer the question.

Attacking the low$_2$ question

Theorem (Lachlan)

Every low$_2$ c.e. set A has a maximal superset.

A is not high iff there is a *true stage enumeration* of A, i.e. there is $\{A_s | s \in \omega\}$ such that if $\overline{A}_s = \{a_0^s < a_1^s < \dots\}$ and $\overline{A} = \{a_0 < a_1 < \dots\}$ then for infinite many s, $a_s^s = a_s$. If $x \in \hat{V}_{0,s}$ at a true stage s then $x \notin A$. Not as strong as access to $\overline{A}$ as before.

Previously our informational questions were of form, is $W \cap \overline{A}$ is infinite, where W is the set of balls which enter a region. Lachlan used the question are there infinite many balls in the region at true stages, i.e. is W^A infinite?

Question

Must every c.e. set A with a true stage enumeration and semilow$_2$ complement have a maximal superset?