Generic Muchnik reducibility

Joseph S. Miller
University of Wisconsin–Madison

(Joint work with Andrews, Schweber, and M. Soskova)

Computability and Complexity Symposium 2017 (Rodfest)
Muchnik reducibility between structures

Definition
If $\mathcal{A}$ and $\mathcal{B}$ are countable structures, then $\mathcal{A}$ is Muchnik reducible to $\mathcal{B}$ (written $\mathcal{A} \preceq_w \mathcal{B}$) if every $\omega$-copy of $\mathcal{B}$ computes an $\omega$-copy of $\mathcal{A}$.

- $\mathcal{A} \preceq_w \mathcal{B}$ can be interpreted as saying that $\mathcal{B}$ is intrinsically at least as complicated as $\mathcal{A}$.

- This is a special case of Muchnik reducibility; it might be more precise to say that the problem of presenting the structure $\mathcal{A}$ is Muchnik reducible to the problem of presenting $\mathcal{B}$.

- Muchnik reducibility doesn’t apply to uncountable structures.

Various approaches have been used to extend computable structure theory beyond the countable:

- Computability on admissible ordinals (aka $\alpha$-recursion theory)
- Computability on separable structures, as in computable analysis
- ...
Generic Muchnik reducibility

Noah Schweber extended Muchnik reducibility to arbitrary structures (see Knight, Montalbán, Schweber):

Definition (Schweber)
If $\mathcal{A}$ and $\mathcal{B}$ are (possibly uncountable) structures, then $\mathcal{A}$ is generically Muchnik reducible to $\mathcal{B}$ (written $\mathcal{A} \leq^*_w \mathcal{B}$) if $\mathcal{A} \leq_w \mathcal{B}$ in some forcing extension of the universe in which $\mathcal{A}$ and $\mathcal{B}$ are countable.

It follows from Shoenfield absoluteness that generic Muchnik reducibility is robust.

Lemma (Schweber)
If $\mathcal{A} \leq^*_w \mathcal{B}$, then $\mathcal{A} \leq_w \mathcal{B}$ in every forcing extension that makes $\mathcal{A}$ and $\mathcal{B}$ countable.

In particular, for countable structures, $\mathcal{A} \leq^*_w \mathcal{B} \iff \mathcal{A} \leq_w \mathcal{B}$. 
Initial example

Definition (Cantor space)
Let $\mathcal{C}$ be the structure with universe $2^{\omega}$ and predicates $P_n(X)$ that hold if and only if $X(n) = 1$.

Observation (Knight, Montalbán, Schweber)
$$\mathcal{C} \equiv^*_w (\mathbb{R}, +, \cdot).$$

To understand this example, say that we take a forcing extension that collapses the continuum.

The Turing degrees from the ground model now form a countable ideal $I$. By absoluteness, this ideal has many of the properties it has in the ground model. It’s a jump ideal and much more.

Let $\mathbb{R}_I$ be the reals in $I$ (the ground model’s version of $\mathbb{R}$). Similarly, let $\mathcal{C}_I$ denote the restriction of $\mathcal{C}$ to sets in $I$ (the ground model’s version of $\mathcal{C}$).
Initial example

Facts

- From a copy of $(\mathbb{R}_I, +, \cdot)$, or even $(\mathbb{R}_I, +, <)$, we can compute an injective listing of the sets in $I$, i.e., one with no repetitions.

- A degree $d$ computes a copy of $C_I$ iff it computes an (injective) listing of the sets in $I$.

This shows that $C_I \leq_w (\mathbb{R}_I, +, <)$. It is even easier to see that $(\mathbb{R}_I, +, <) \leq_w (\mathbb{R}_I, +, \cdot)$.

Therefore, $C \leq^*_w (\mathbb{R}, +, <) \leq^*_w (\mathbb{R}, +, \cdot)$.

Question (Knight, Montalbán, Schweber)

Is $(\mathbb{R}, +, \cdot) \leq^*_w C$?

No! This was answered by Igusa and Knight, and independently (though later) by Downey, Greenberg, and M.
Facts about $\mathcal{C}$ and $\mathcal{B}$

Definition (Baire space)

Let $\mathcal{B}$ be the structure with universe $\omega^\omega$ and, for each finite string $\sigma \in \omega^{<\omega}$, a predicate $P_\sigma(f)$ that holds if and only if $\sigma < f$.

The following facts were proved by Igusa, Knight; Downey, Greenberg, M.; Igusa, Knight, Schweber; Andrews, Knight, Kuyper, Lempp, M., Soskova.

- $\mathcal{B} \equiv_w^* (\mathbb{R}, +, <) \equiv_w^* (\mathbb{R}, +, \cdot)$. This degree also contains every closed/continuous expansion of $(\mathbb{R}, +, \cdot)$.

- $\mathcal{C} <_w^* \mathcal{B}$.

- $\mathcal{C}' \equiv_w^* \mathcal{B}$.

- The closed/continuous expansions of $\mathcal{C}$ lie in the interval between $\mathcal{C}$ and $\mathcal{B}$.

Question

Is there a generic Muchnik degree strictly between $\mathcal{C}$ and $\mathcal{B}$?
Definability and post-extension complexity

It is going to be important to understand the complexity of definable sets both before and after the forcing extension.

**Definition**
We say that a relation $R$ on a structure $\mathcal{M}$ is $\Sigma^c_n(\mathcal{M})$ if it is definable by a computable $\Sigma_n L_{\omega_1 \omega}$ formula with finitely many parameters.

**Theorem (Ash, Knight, Manasse, Slaman; Chisholm)**
If $\mathcal{M}$ is countable, then $R$ is $\Sigma^c_n(\mathcal{M})$ if and only if it is relatively intrinsically $\Sigma^0_n$, i.e., its image in any $\omega$-copy of $\mathcal{M}$ is $\Sigma^0_n$ relative to that copy.

Computable objects and satisfaction on a structure are absolute, so:

**Corollary**
A relation $R$ is $\Sigma^c_n(\mathcal{M})$ if and only if it is relatively intrinsically $\Sigma^0_n$ in any/every forcing extension that makes $\mathcal{M}$ countable.
Definability and pre-extension complexity

In structures like $C$ and $B$, we can also measure the complexity of $\Sigma^c_n(\mathcal{M})$ relations in topological terms.

The calculation depends on the structure:

\[
\begin{array}{cccccc}
B & \Sigma^c_2 & \Sigma^c_3 & \Sigma^c_4 & \Sigma^c_5 & \Sigma^c_6 & \ldots \\
\Sigma^1_1 & \Sigma^1_2 & \Sigma^1_3 & \Sigma^1_4 & \Sigma^1_5 & \ldots \\
C & \Sigma^0_2 & \Sigma^1_1 & \Sigma^1_2 & \Sigma^1_3 & \Sigma^1_4 & \ldots \\
\end{array}
\]

- These bounds are sharp, e.g., every $\Sigma^1_1$ relation on $B$ is $\Sigma^c_2(B)$.
- The “lost quantifiers” correspond to the first order quantifiers needed in the normal form for $\Sigma^1_n$ relations with function/set quantifiers.
- This leads to an easy (and essentially different) separation between the generic Muchnik degrees of $C$ and $B$. 
Differentiating $\mathcal{C}$ and $\mathcal{B}$ with a linear order

**Lemma**
There is a linear order $\mathcal{L}$ such that $\mathcal{L} \leq_w^{*} \mathcal{B}$ but $\mathcal{L} \not\leq_w^{*} \mathcal{C}$.

**Proof Idea**
For $X \subseteq \mathcal{C}$, we define a linear order $\mathcal{L}_X$ that codes $X$. It is essentially a shuffle sum of delimited $\zeta$-representations of all elements of Cantor space along with markers for the sequences not in $X$.

It is designed so that:

- If $X$ is $\Pi^c_3(\mathcal{B})$, then $\mathcal{L}_X \leq_w^{*} \mathcal{B}$,
- If $\mathcal{L}_X \leq_w^{*} \mathcal{C}$, then $X$ is $\Sigma^c_4(\mathcal{C})$.

Now take $X \subseteq \mathcal{C}$ to be $\Pi^1_2$ but not $\Sigma^1_2$. By the analysis on the previous slide:

- $X$ is $\Pi^c_3(\mathcal{B})$, so $\mathcal{L}_X \leq_w^{*} \mathcal{B}$,
- $X$ is not $\Sigma^c_4(\mathcal{C})$, so $\mathcal{L}_X \not\leq_w^{*} \mathcal{C}$.
A degree strictly between $\mathcal{C}$ and $\mathcal{B}$

**Lemma**
There is a linear order $\mathcal{L}$ such that $\mathcal{L} \leq_w^* \mathcal{B}$ but $\mathcal{L} \leq^*_w \mathcal{C}$.

But linear orders are bad at coding:

**Lemma**
If $\mathcal{L}$ is a linear order, then $\mathcal{B} \leq^*_w \mathcal{C} \sqcup \mathcal{L}$.

Following the Downey, Greenberg, M. proof that $\mathcal{B} \leq^*_w \mathcal{C}$, we show that a generic countable presentation of $\mathcal{C} \sqcup \mathcal{L}$ does not compute a copy of $\mathcal{B}$. The key fact used about linear orders is that their $\sim_2$-equivalence classes are tame (Knight 1986).

Now let $\mathcal{M} = \mathcal{C} \sqcup \mathcal{L}$, where $\mathcal{L}$ is the linear order from the first lemma.

**Corollary**
There is a structure $\mathcal{M}$ such that $\mathcal{C} <^*_w \mathcal{M} <^*_w \mathcal{B}$.

Great! But...not the most satisfying example.
What kind of example would we like?

The initial attempts to find an intermediate degree involved natural expansions of $\mathcal{C}$, but without success. For example:

- $(\mathcal{C}, \oplus) \equiv^*_w (\mathcal{C}, \sigma) \equiv^*_w \mathcal{B}$, where $\sigma$ is the shift operator on $2^\omega$.
- $(\mathcal{C}, \subseteq) \equiv^*_w (\mathcal{C}, \triangle) \equiv^*_w \mathcal{C}$.

Another approach would be to expand $\mathcal{C}$ with sufficiently generic relations. Greenberg, Igusa, Turetsky, and Westrick tried a version of this that involved adding infinitely many unary relations.

In both cases, we considered expansions of $\mathcal{C}$.

Open Question

Is there an expansion of $\mathcal{C}$ that is strictly between $\mathcal{C}$ and $\mathcal{B}$?
Expansions of $\mathcal{C}$ above $\mathcal{B}$

Let $\mathcal{M} = (\mathcal{C}, \text{Stuff})$ be an expansion of $\mathcal{C}$. First, we want a criterion that guarantees that $\mathcal{M} \geq^*_w \mathcal{B}$.

- If the set $\mathcal{F} \subset 2^\omega$ of sequences with finitely many ones is $\Delta^c_1(\mathcal{M})$, i.e., computable in every $\omega$-copy of $\mathcal{M}$, then $\mathcal{M} \geq^*_w \mathcal{B}$.
  - Why? There is a natural bijection between $\mathcal{B}$ and $\mathcal{C} \setminus \mathcal{F}$.

- If $\mathcal{F}$ is $\Delta^c_2(\mathcal{M})$, then $\mathcal{M} \geq^*_w \mathcal{B}$.
  - Add a little injury.
  - This is how we show, for example, that $(\mathcal{C}, \oplus) \geq^*_w \mathcal{B}$.

- If any countable dense set is $\Delta^c_2(\mathcal{M})$, then $\mathcal{M} \geq^*_w \mathcal{B}$.

- If there is a perfect set $\mathcal{P} \subseteq \mathcal{C}$ with a countable dense $\mathcal{Q} \subseteq \mathcal{P}$ that is $\Delta^c_2(\mathcal{M})$, then $\mathcal{M} \geq^*_w \mathcal{B}$. 
Expansions of $\mathcal{C}$ above $\mathcal{B}$

- If there is a perfect set $\mathcal{P} \subseteq \mathcal{C}$ with a countable dense $\mathcal{Q} \subseteq \mathcal{P}$ that is $\Delta^c_2(\mathcal{M})$, then $\mathcal{M} \geq^*_w \mathcal{B}$.

**Lemma**

If $\mathcal{M} \leq^*_w \mathcal{B}$ and $R \subseteq \mathcal{C}$ is $\Delta^c_2(\mathcal{M})$, then it is $\Delta^c_2(\mathcal{B})$, i.e., Borel.

**Lemma (Hurewicz)**

If $R \subseteq \mathcal{C}$ is Borel but not $\Delta^0_2$, then there is a perfect set $\mathcal{P} \subseteq \mathcal{C}$ such that either $\mathcal{P} \cap R$ or $\mathcal{P} \setminus R$ is countable and dense in $\mathcal{P}$.

Putting it all together (and noting that arity doesn’t matter):

**Lemma**

If $\mathcal{M} \leq^*_w \mathcal{B}$ is an expansion of $\mathcal{C}$ and $R \subseteq \mathcal{C}^n$ is $\Delta^c_2(\mathcal{M})$ but not $\Delta^0_2$, then $\mathcal{M} \geq^*_w \mathcal{B}$. 
Tameness and dichotomy

In the contrapositive (and using the fact that $\Delta_0^2 = \Delta_2^c(\mathcal{C})$):

**Tameness Lemma**
If $\mathcal{M} <^*_w \mathcal{B}$ is an expansion of $\mathcal{C}$, then $\Delta_2^c(\mathcal{M}) = \Delta_2^c(\mathcal{C})$.

**Dichotomy Theorem for Closed Expansions**
If $\mathcal{M} \leq^*_w \mathcal{B}$ is an expansion of $\mathcal{C}$ by closed relations (and/or continuous functions), then either $\mathcal{M} \equiv^*_w \mathcal{C}$ or $\mathcal{M} \equiv^*_w \mathcal{B}$.

**Proof Idea**
For a tuple $\overline{X} \subset \mathcal{C}$, let $p(\overline{X})$ be the (code for the) complete positive $\Sigma_1(\mathcal{M})$ type of $\overline{X}$. The relation that holds only on tuples of the form $(\overline{X}, p(\overline{X}))$ is $\Delta_2^c(\mathcal{M})$.

If it is not $\Delta_2^c(\mathcal{C})$, then $\mathcal{M} \geq^*_w \mathcal{B}$.

If it is $\Delta_2^c(\mathcal{C})$, then a delicate injury argument can be used to prove that $\mathcal{M} \leq^*_w \mathcal{C}$.
Another dichotomy result

Combined with work of Greenberg, Igusa, Turetsky, and Westrick:

**Dichotomy Theorem for Unary Expansions**

If $\mathcal{M} \leq_{w}^* \mathcal{B}$ is an expansion of $\mathcal{C}$ by countably many unary relations, then either $\mathcal{M} \equiv_{w}^* \mathcal{C}$ or $\mathcal{M} \equiv_{w}^* \mathcal{B}$.

- If $\mathcal{M}$ is an expansion of $\mathcal{C}$ by finitely many $\Delta^0_2$ unary relations, then $\mathcal{M} \leq_{w}^* \mathcal{C}$. This is a fairly simple finite injury argument.

- Expansions by infinitely many closed unary relations need not be below $\mathcal{C}$: For $\sigma \in 2^{<\omega}$, let $U_\sigma$ hold only on $\sigma0^\omega$. Then the set of sequences with finitely many ones is $\Sigma^c_1(\mathcal{C}, \{U_\sigma\}_{\sigma \in 2^{<\omega}})$.

- Greenberg, et al. supplied the right condition distinguishing the cases, and one direction of the proof.

The dichotomy results kill off a lot of possible natural (and many unnatural) examples of expansions.
Recent progress

Based partly on conversations with Turetsky and Gura, I am pretty sure the following is true.

Using Marker extensions, we can get structures with the following “complexity profiles”:

\[
\begin{array}{ccccccc}
\Sigma_2^c & \Sigma_3^c & \Sigma_4^c & \Sigma_5^c & \Sigma_6^c & \ldots \\
\hline
\Sigma_1^1 & \Sigma_2^1 & \Sigma_3^1 & \Sigma_4^1 & \Sigma_5^1 & \ldots \\
\Sigma_2^0 & \Sigma_2^1 & \Sigma_3^1 & \Sigma_4^1 & \Sigma_5^1 & \ldots \\
\Sigma_2^0 & \Sigma_1^1 & \Sigma_2^1 & \Sigma_4^1 & \Sigma_5^1 & \ldots \\
\Sigma_2^0 & \Sigma_1^1 & \Sigma_2^1 & \Sigma_3^1 & \Sigma_4^1 & \ldots \\
\vdots & & & & & \\
\Sigma_2^0 & \Sigma_1^1 & \Sigma_2^1 & \Sigma_3^1 & \Sigma_4^1 & \ldots \\
\end{array}
\]

- Again, these bounds are sharp.
- \( C <^*_w \cdots <^*_w M_3 <^*_w M_2 <^*_w M_1 <^*_w B \).
Open questions

1. Can an expansion of $C$ be strictly between $C$ and $B$? (In particular, the non-unary $\Delta^0_2$ case is open.)

2. Are the degrees of $M_1, M_2, M_3, \ldots$ the only degrees strictly between $C$ and $B$?

3. Are there incomparable degrees between $C$ and $B$?

These questions are related. For example:

**Fact.** Any Borel expansion of $C$ that is not above $B$ has the same complexity profile as $C$. So a positive answer to 1 gives a negative answer to 2.

We have focused on $C$ and $B$ (and a couple of other degrees). What else are generic Muchnik degrees good for?
Thank you.

And thanks to Rod for being a great friend and mentor!