My work with Rod 1995-2001

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Degree structures

Back in the days before everyone started using the beamer package, degree structures were studied even more intensely than now.

My work with Rod 1995-2001 was centred around degree structures.

We looked at reducibilities other than Turing:

• $\leq_Q$,
• $\leq_m^p$, $\leq_T^p$ (polynomial time $m$- and Turing),
• $\leq_S$ (Solovay).
Rod, I and co-authors studied three very different degree structures.

- The $Q$ degrees of r.e. sets, with LaForte

- polynomial time many-one and Turing degrees of exponential time sets of strings

- Solovay degrees of left-r.e. reals, with Hirschfeldt.
I. \(\mathcal{Q}\)-degrees of r.e. sets

\(\mathcal{Q}\)-reducibility was defined by Tennenbaum (according to Rogers). The “\(\mathcal{Q}\)” stands for “quasi”. In the Turing case we have for r.e. sets \(A \leq_T B \iff \exists R \text{ r.e. } \forall y [y \in \overline{A} \iff \exists z [(y, z) \in R \& D_z \subseteq B]].\)

For \(\mathcal{Q}\)-reducibility, the reduction procedure is allowed at most one negatively answered oracle question. Thus, each \(D_z\) is a singleton. Now let \(W_g(y) = \bigcup_{(y, z) \in R} D_z\) for computable \(g\):

**Definition 1** For \(A, B \subseteq \mathbb{N}\), let \(A \leq_{\mathcal{Q}} B \iff \exists g \text{ computable } \forall y [y \in A \iff W_g(y) \subseteq B].\)

For r.e. sets \(A \leq_{\mathcal{Q}} B \Rightarrow A \leq_T B.\)

\(\mathcal{Q}\)-reducibility was used for a structural solution to Post’s problem (Marchenkov), and for the complexity of word problems of groups.
Meets and joins in $\mathcal{R}_Q$

Results below are from Downey, N, LaForte, *Computably enumerable sets and quasi-reducibility*, APAL 95.1 (1998): 1-35.

First note that $\mathcal{R}_Q$ is an upper semilattice where the sup of degrees of $A, B$ is the degree of $A \oplus B$.

**Theorem 2** There is a minimal pair of r.e. $Q$-degrees that have the same Turing degree.

The proof uses a pinball machine model.

**Theorem 3** There is a meet-irreducible r.e. $Q$-degree outside any nontrivial upper cone.
The hardest result answers a question of Ishmukametov.

**Theorem 4** *The r.e. $Q$-degrees are dense.*

Given r.e. $B <_Q A$, we want to build r.e. $C$ such that $B <_Q B \oplus C <_Q A$. The proof is harder than in the Turing case. For instance, the usual permitting technique for $C \leq_T A$ doesn’t yield a $Q$-reduction. We use a tree of strategies.

Truth table reducibility (not dense) and $Q$-reducibility are incomparable reducibilities on the r.e. sets; the result shows that $Q$-reducibility is in a sense closer to Turing.
Undecidability of $\text{Th}(\mathcal{R}_Q)$

**Theorem 5** *The first-order theory of $\mathcal{R}_Q$ is undecidable.*

This is proved by encoding with parameters any given computable partial order $(\mathbb{N}, \preccurlyeq)$.

- The domain is represented by a Slaman-Woodin set $\{G_i\}$ (the sets below $C$ of minimal degree cupping $P$ above $Q$).
- Build a further parameter $L$ with $i \preceq k \iff G_i \leq_Q G_k \oplus L$. 
Some later work on $Q$-degrees

- Affatato, Kent and Sorbi 2007: paper on $s$-degrees (singleton reducibility). This is a restricted version of $e$-reducibility. Note that $\overline{A} \leq_s \overline{B} \iff A \leq_Q B$.
  They show the $\Sigma^0_2$ and the $\Pi^0_1$ $s$-degrees are undecidable, using “exact degree theorems”.

- Arslanov, Baturshin and Omanadze (2007) work on $n$-r.e. $Q$-degrees.
  They also prove that there is a noncappable incomplete r.e. $Q$-degree.
II: subrecursive degree structures

Let $\Sigma$ be alphabet, $X, Y \subseteq \Sigma^*$ languages over $\Sigma$.

$X \leq_{pm}^P Y \iff \exists f \in P \ [X = f^{-1}(Y)]$

$X \leq_{PT}^P Y \iff \exists$ polynomial time bounded oracle TM which computes $X$ with oracle $Y$.

Ladner (1975) proved that the degree structures induced on the computable languages are dense.

**Theorem 6 (Downey and N., JCSS 2003)** The polynomial time many-one and Turing degrees of languages in $\text{DTIME}(2^n)$ have an undecidable theory.

Instead of $2^n$, one can take any nondecreasing time constructible function $h : \omega \to \omega$ such that $P \subset \text{DTIME}(h)$. E.g. $h(n) = n^{\log \log n}.$
Undecidable ideal lattices

• A structure $(\mathbb{N}, \preceq, \wedge, \vee)$ is a $\Sigma^0_k$-boolean algebra if $\preceq$ is $\Sigma^0_k$, and the operations $\wedge, \vee$ are recursive in $\emptyset^{(k-1)}$.

• A $\Sigma^0_k$-boolean algebra $\mathcal{B}$ is called **effectively dense** if there is a function $F \leq_T \emptyset^{(k-1)}$ such that

  $\forall x \ [F(x) \preceq x]$ and

  $\forall x \neq 0 \ [0 \prec F(x) \prec x]$.

• For a $\Sigma^0_k$-boolean algebra $\mathcal{B}$, let $\mathcal{I}(\mathcal{B})$ be the lattice of $\Sigma^0_k$-ideals of $\mathcal{B}$ with $\cap$ and $\lor$ as operations.

**Theorem 7 (N, Trans. AMS 2000)** Suppose $\mathcal{B}$ is an effectively dense $\Sigma^0_k$-Boolean algebra. Then $\text{Th}(\mathbb{N}, +, \cdot) \equiv_m \text{Th}(\mathcal{I}(\mathcal{B}))$.

It is much easier to show that $\text{Th}(\mathcal{I}(\mathcal{B}))$ is hereditarily undecidable [N., Bull. LMS 1997].
Undecidability via coding $\mathcal{I}(\mathcal{B})$

It is often natural to interpret with parameters $\mathcal{I}(\mathcal{B})$ in a structure. This shows that the structure has an undecidable theory. If no parameters are needed, it yields an interpretation of $\text{Th}(\mathbb{N}, +, \cdot)$ in the theory of the structure.

- Intervals of $\mathcal{E}^*$ that are not Boolean algebras; no parameters needed (N., 1997)

- Computable sets with parameterized reducibilities (Coles, Downey, Sorbi, year?).

- Solovay degrees of left-r.e. reals (Downey, Hirschfeldt and LaForte, JCSS, 2007). Details later.
Supersparse sets in complexity theory

We will apply the method of coding $\mathcal{I}(\mathcal{B})$ in the proof that polytime degrees of $\text{DTIME}(h)$ have an undecidable theory. Here $k = 2$, because $\leq^p_m$ and $\leq^p_T$ are $\Sigma^0_2$-relations on such a class.

**Definition 8 (Ambos-Spies 1986)** Let $f : \omega \to \omega$ be a strictly increasing, time constructible function. We say that a language $A \subseteq \{0^f(k) : k \in \omega\}$ is super sparse via $f$ if

"$0^f(k) \in A$?" can be determined in time $O(f(k + 1))$.

Supersparse sets exist in the time classes we are interested in.

**Lemma 9 (Ambos-Spies 1986)** Suppose that $h : \omega \to \omega$ is a nondecreasing time constructible ("nice") function with $P \subset \text{DTIME}(h)$. Then there is a super sparse language $A \in \text{DTIME}(h) - P$. 


Interpreting $\mathcal{I}(\mathcal{B})$ in $[0, a]$ for super sparse $a$

Now let $f, h$ be time constructible functions as above, let $A \in \text{DTime}(h) - P$ be supersparse via $f$, and $a$ be its degree. Ambos-Spies has shown that $[0, a]$ is a distributive lattice that does not depend on the reducibility.

- Each complemented element in $[0, a]$ is the degree of a splitting $A \cap R$, where $R$ is polytime.
- This implies that the algebra $\mathcal{B}$ of complemented elements is $\Sigma^0_2$ and effectively dense.
- Downey and N. showed that for each ideal $I$ in $\mathcal{I}(\mathcal{B})$, there is $c_I \leq a$ such that $x \in I \iff x \leq c_I$ for each $x \leq a$.

So one can interpret $\mathcal{I}(\mathcal{B})$ in $[0, a]$ without parameters (and hence in $\text{DTime}(h)$ with parameter $a$).
III. Solovay reducibility on left-r.e. reals

A real $\alpha$ is left-r.e. if there is a non-decreasing effective sequence $(\alpha_s)$ of rationals converging to $\alpha$.

We will use $\alpha, \beta, \gamma$ to denote left-r.e. reals. We think of them as equipped with an effective sequence of rationals of this kind.

Example of a left-r.e. real: The halting probability of a fixed universal prefix-free machine $U$

$$\Omega = \sum\{2^{|\sigma|} : U(\sigma) \downarrow\}.$$
Solovay reducibility

Solovay (1975) introduced a reducibility $\leq_S$ to compare the “randomness content” of left-r.e. reals.

$\beta \leq_S \alpha \iff$

$\exists C \in \mathbb{Q} \ \exists f \text{ computable increasing} \ \forall s \ [\beta - \beta_{f(s)} \leq C(\alpha - \alpha_s)].$

He proved that $\beta \leq_S \alpha \Rightarrow \exists c \forall n \ K(\beta \upharpoonright n) \leq K(\alpha \upharpoonright n) + c.$

$\Omega$ is $\leq_S$–complete. Kucera and Slaman (2001) showed that, for left-r.e. reals,

$\leq_S$–complete $\iff$ ML-random.

**Fact 10** $\beta \leq_S \alpha \iff \exists C \text{ rational} \ \exists \gamma \ C(\beta + \gamma) = \alpha.$

**Fact 11** $\beta \leq_S \alpha \Rightarrow \beta \leq_T \alpha.$ But $\leq_S$ is incomparable with $\leq_{wtt}.$
Algebraic properties

Results below are from Downey, Hirschfeldt, Nies, *Randomness, computability, and density*, Siam J. Computing 31, 2002.

Let $S$ be the degree structure induced on the r.e. reals. We investigate the algebraic properties of $S$.

$S$ is an upper semilattice (u.s.l.) where the sup is given by the usual addition.

Recall that an u.s.l. is distributive if it satisfies

$$a \leq b \lor c \Rightarrow a = \tilde{b} \lor \tilde{c} \text{ for some } \tilde{b} \leq b, \tilde{c} \leq c.$$ 

**Proposition 12** $S$ is a distributive u.s.l.

Among the common degree structures on r.e. sets, $R_m$ (many-one) and $R_{wtt}$ (weak truth-table) are distributive.

Is $S$ more like $R_{wtt}$, or more like $R_m$?
Using standard coding and preservation strategies, we obtain upward density.

**Theorem 13** Let $\gamma < S \Omega$. Then there is $\beta$ such that $\gamma < S \beta < S \Omega$.

If $\alpha < S \Omega$, we prove that in a sense any sequence for $\Omega$ converges much slower than one for $\alpha$. This gives combined splitting and density below $\alpha$.

**Theorem 14** Let $\gamma < S \alpha < S \Omega$. There are $\beta^0$ and $\beta^1$ such that $\gamma < S \beta^0, \beta^1 < S \alpha$ and $\beta^0 + \beta^1 = \alpha$.

Combining the two, we obtain a (non-uniform) proof of density.

$S$ shares this property with $R_{wtt}$. 
**Random left-r.e. reals**

**Fact 15** *If one of $\alpha, \beta$ is ML-random, then $\gamma = \alpha + \beta$ is ML random.*

By contraposition suppose that $\gamma$ is not ML-random. So $\gamma \in \bigcap G_m$ for a ML-test $(G_m)$, where $\lambda G_m \leq 2^{-m-1}$. Build a ML-test $(H_m)$ for $\alpha$: At stage $s$, if $\gamma_s \in I$ where $I = [x, y)$ is a maximal subinterval of $G_{m,s}$, then put the interval

$$J = [x - \beta_s - (y - x), y - \beta_s]$$

into $H_m$. (Note that $J$ is twice as long as $I$.)

A similar fact *fails* for left-r.e. reals and weaker randomness notions. The opposite was announced (wrongly) during the talk.

**Fact 16 (with Miyabe and Stephan, 2017)** *There is $\alpha$ partial computably random and $\beta$ such that $\alpha + \beta$ is not Kurtz random.*
Random left-r.e. reals

Now for the converse for ML-randomness.

**Theorem 17** If $\alpha + \beta$ is ML-random, then one of $\alpha, \beta$ is ML random.

(Several years after our paper appeared in 2001, Kucera pointed out that this was claimed without proof by Demuth\textsuperscript{a}.

Using the Kucera and Slaman Theorem that any random left-r.e. real is $\leq S$-complete, this implies

**Corollary 18** In $S$, the greatest element is join irreducible.

$S$ shares this property with $R_m$.

\textsuperscript{a}Constructive pseudonumbers, Comment. Math. Univ. Carolinae, vol. 16 (1975), pp. 315 - 331, Russian)
Later work

- Downey, Hirschfeldt and LaForte, *Undecidability of the structure of the Solovay degrees of c.e. reals* (2002) uses the method of coding $I(\mathcal{B})$. Similar to the complexity case, they build an r.e. set $A$ such that all complemented elements below are given by r.e. splittings.

- Downey, Hirschfeldt and LaForte, *Randomness and reducibility*, JCSS, 2007 (also D-H book): proof of results such as density in a more general axiomatic setting; works for $\leq S, \leq C, \leq K, \leq rK$ but not $\leq_{sw}$.

Additive cost functions

Not a lot has happened on the structure of Solovay degrees in recent years. However, I used $\leq_S$ in the paper “Calculus of cost functions” (to appear in “The Incomputable”).

For a r.e. real $\beta$ with a given approximation let $c_\beta(x, s) = \beta_s - \beta_x$.

**Proposition 19** $c_\alpha$ implies $c_\beta$ for some approximations of $\alpha, \beta$

$\iff \beta \leq_S \alpha$.

E.g. $c_\Omega$ is the strongest additive cost function. Obeying it characterises the $K$-trivials.

**Question 20** Find $\beta$ such that the $\Delta^0_2$ sets obeying $c_\beta$ form a proper Turing ideal different from the $K$-trivials.
More open questions

**Question 21** Do the degree structures considered above interpret true arithmetic?

**Question 22** Suppose $a \neq 0$ is a polytime $m$ (or Turing) degree. Is $Th[0, a]$ undecidable?

**Question 23** How can we distinguish incomplete Solovay degrees of left-r.e. reals? For instance, are there two non-isomorphic initial segments strictly below $\Omega$?

Also: study the Solovay degrees of left-r.e. Schnorr randoms.