Irrationality Exponents and Effective Hausdorff Dimension

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Ralph Waldo Emerson on the purpose of life:

It is to be useful, to be honorable, to be compassionate, to have it make some difference that you have lived and lived well.

Cheers Rod on an exemplary mathematical life:

- Practitioner
- Expositor
- Mentor
- Leader
Abstract

Suppose $a \geq 2$ and $b \in [0, 2/a]$.

- (Generalization of Jarník 1929 and Besicovitch 1934) There is a Cantor-like set with Hausdorff dimension equal to $b$ such that, with respect to its uniform measure, almost all real numbers have irrationality exponent equal to $a$.

- There is a Cantor-like set such that, with respect to its uniform measure, almost all real numbers have effective Hausdorff dimension equal to $b$ and irrationality exponent equal to $a$.

In each case, we obtain the desired set as a distinguished path in a tree of Cantor sets.
Hausdorff Dimension

For a set of real numbers $X$ and a non-negative real number $s$ the $s$-dimensional Hausdorff measure of $X$ is defined by

$$\lim_{\varepsilon \to 0} \inf \left\{ \sum_{j \geq 1} r_j^s : \text{there is a cover of } X \text{ by balls with radii } (r_j : j \geq 1) \text{ and } \forall j (r_j < \varepsilon) \right\}.$$ 

The Hausdorff dimension of $X$ is the infimum of the set of non-negative reals $s$ such that the $s$-dimensional Hausdorff measure of $X$ is zero.
Effective Hausdorff Dimension of $\xi \in 2^\mathbb{N}$

**Definition**

The *effective Hausdorff dimension* of a real number $\xi$ is the infimum of the set of $t$ such that there is a $c$ for which there are infinitely many $\ell$ such that the prefix-free Kolmogorov complexity of the first $\ell$ digits in the binary expansion of $\xi$ is less than $t \cdot \ell + c$.

Heuristic: The effective Hausdorff dimension of a real number $\xi$ is the infimum of the algorithmic compression factors of the initial segments of the binary expansion of $\xi$.

- Computable real numbers have effective dimension 0.
- Random real numbers have effective dimension 1.
- The set of real numbers with effective Hausdorff dimension $b$ has Hausdorff dimension $b$.

There is an equivalent formulation using effectively presented covers.
Irrationality Exponent

**Definition (originating with Liouville 1855)**

For a real number $\xi$, the *irrationality exponent of* $\xi$ is the least upper bound of the set of real numbers $a$ such that

$$0 < \left| \xi - \frac{p}{q} \right| < \frac{1}{q^a}$$

is satisfied by an infinite number of integer pairs $(p, q)$ with $q > 0$.

- When $a$ is large and $0 < \left| \xi - \frac{p}{q} \right| < \frac{1}{q^a}$, then $p/q$ is a good approximation to $\xi$ when considered in the scale of $1/q$.
- The irrationality exponent of $\xi$ is an indicator for how well $\xi$ can be approximated by rational numbers (a linear version of Kolmogorov complexity).
Examples

- Random real numbers have irrationality exponent equal to 2.
- (Roth 1955) Irrational algebraic real numbers have irrationality exponent equal to 2.
- Liouville numbers are those with infinite irrationality exponent—these were the first examples of transcendental numbers.

Example

For $a \geq 2$, $\{\xi : \xi \text{ has irrationality exponent } a\}$ has Hausdorff dimension less than or equal to $2/a$. 
Consequences of Irrationality Exponent for Effective Dimension

Remark

*If* $\xi$ *has irrationality exponent equal to* $a$, *then* $\xi$ *has effective Hausdorff dimension less than or equal to* $2/a$:

Proof

- Say that $|p/q - \xi| < 1/q^a$.
- Need $2 \cdot \log_2 q$ bits to specify $p$ and $q$.
- Obtain $a \cdot \log_2 q$ bits in the binary expansion of $\xi$.
- \[ \frac{2 \cdot \log_2 q}{a \cdot \log_2 q} = \frac{2}{a}. \]
No Other Consequences

The second result mentioned earlier has the following corollary.

**Theorem (Becher, Reimann and Slaman)**

For every \( a \geq 2 \) and every \( b \) in \([0, 2/a]\), there is a real number \( \xi \) such that \( \xi \) has irrationality exponent \( a \) and effective Hausdorff dimension \( b \).
The Jarník-Besicovitch Theorem

Theorem (Jarník 1929 and Besicovitch 1934)

For every real number $a$ greater than or equal to 2, the set of numbers with irrationality exponent equal to $a$ has Hausdorff dimension exactly equal to $\frac{2}{a}$.

As mentioned earlier, it is a direct application of the definitions to show that the Hausdorff dimension of the set of numbers with irrationality exponent $a$ is less than or equal to $\frac{2}{a}$. The other inequality comes from an early application of fractal geometry.
Jarník’s Fractal

For each real number $a$ greater than 2, Jarník gave a Cantor-like construction of a fractal $J$ contained in $[0, 1]$ of Hausdorff dimension $2/a$ such that the uniform measure $\nu$ on $J$ satisfies the following:

- Every element of $J$ has irrationality exponent greater than or equal to $a$.
- For all $b$ greater than $a$, the set of numbers with irrationality exponent greater than or equal to $b$ has $\nu$-measure equal to 0.

Let $(M_i : i \in \mathbb{N})$ be a rapidly increasing sequence of natural numbers.
Define $(E_i : i \in \mathbb{N})$ as follows.

- $E_0 = [0, 1]$
- For $i > 0$, let

$$E_i = \bigcup \left\{ \left[ \frac{p}{q} - \frac{1}{q^a}, \frac{p}{q} + \frac{1}{q^a} \right] : 0 < p < q, M_i \leq q \leq 2M_i, q \text{ prime}, \right\}$$

$$[p/q - 1/q^a, p/q + 1/q^a] \subset E_{i-1}$$

Let $J = \bigcap_{i \in \mathbb{N}} E_i$. 
The Mass Distribution Principle

Every element of $J$ has irrationality exponent less than or equal to $a$, so the Hausdorff dimension of $J$ is less than or equal to $2/a$.

Show that $J$ has Hausdorff dimension at least $2/a$ by applying the following fact for the uniform measure $\mu$ on $J$.

**Theorem (Mass Distribution Principle)**

*Let $\nu$ be a finite measure, $d$ a positive real number and $X$ a set with Hausdorff dimension less than $d$. Suppose that there is a positive real number $C$ such that for every interval $I$, $\nu(I) < C \vert I \vert^d$. Then $\nu(X) = 0$.***
Modifying $J$ – Version 1

For $a \geq 2$ and $b \in [0, 2/a]$, there is a Cantor-like set with Hausdorff dimension equal to $b$ such that, with respect to its uniform measure, almost all real numbers have irrationality exponent equal to $a$.

Find $J_1 \subset J$ by thinning the levels of $J$, either by using fewer primes or by using one prime and fewer intervals $[p/q - 1/q^a, p/q + 1/q^a]$ and let $\mu_1$ be the uniform measure on $J_1$.

▶ Ensure that the intervals from $E_i$ which are retained to form $J_1$ provide the covers needed to show that $J_1$ has Hausdorff dimension less than or equal to $b$.

▶ Ensure the MDP for $\mu_1$ with exponent $b$, and thereby ensure that $J_1$ has Hausdorff dimension exactly equal to $b$.

▶ Ensure that $\mu_1$-almost all elements of $J_1$ have irrationality exponent equal to $a$ by choosing from among all possible thinnings the one that minimizes the frequency of occurrences of rational approximation with exponent greater than $a$. 
For \( a \geq 2 \) and \( b \in [0, 2/a] \), there is a Cantor-like set such that, with respect to its uniform measure, almost all elements in the set have effective Hausdorff dimension equal to \( b \) and irrationality exponent equal to \( a \).

Find \( J_2 \subset J \) by thinning the levels of \( J \), either by using fewer primes or by using one prime and fewer intervals \([p/q - 1/q^a, p/q + 1/q^a]\) and let \( \mu_2 \) be the uniform measure on \( J_1 \).

- Stratify the construction of \( J_2 \) into extended computable blocks of dimension close to \( b \), thereby producing for each element of \( J_2 \) instances of algorithmic compression approaching \( b \) and ensuring that \( \mu_2 \)-almost every element of \( J_2 \) has effective Hausdorff dimension less than or equal to \( b \).

- Ensure the MDP for \( \mu_2 \) with exponent \( b \). Thus, for \( d < b \), the set of real numbers with effective Hausdorff dimension equal to \( d \) is a \( \mu_2 \)-null set and so \( \mu_2 \)-almost every element of \( J_2 \) has effective Hausdorff dimension exactly \( b \).

- Ensure that \( \mu_2 \)-almost all elements of \( J_2 \) have irrationality exponent equal to \( a \) as before.
The End