There are no maximal dre wtt-degrees

Wu Guohua

Nanyang Technological University

01-07-2017
Turing Jump and High/Low Hierarchy

- $K^A = \{e : \Phi_e^A(e) \downarrow\}$, relativised Halting problem
Turing Jump and High/Low Hierarchy

- $K^A = \{ e : \Phi_e^A(e) \downarrow \}$, relativised Halting problem

- $K^A$, is called the Turing Jump of $A$, denoted as $A'$. 
Turing Jump and High/Low Hierarchy

▶ $K^A = \{ e : \Phi^A_e(e) \downarrow \}$, relativised Halting problem

▶ $K^A$, is called the Turing Jump of $A$, denoted as $A'$.

▶ $'$ is increasing, and has range $\geq 0'$. 

Turing Jump and High/Low Hierarchy

- $K^A = \{ e : \Phi^A_e(e) \downarrow \}$, relativised Halting problem

- $K^A$, is called the Turing Jump of $A$, denoted as $A'$.

- $'$ is increasing, and has range $\geq 0'$.

Jump Inversion Theorems:

- Friedberg, Shoenfield, Sacks
Turing Jump and High/Low Hierarchy

- \( K^A = \{ e : \Phi^A_e(e) \downarrow \} \), relativised Halting problem

- \( K^A \), is called the Turing Jump of \( A \), denoted as \( A' \).

- \( ' \) is increasing, and has range \( \geq 0' \).

Jump Inversion Theorems:

Friedberg, Shoenfield, Sacks

- Low degrees and High degrees, High/Low hierarchy

- Sacks: There exist intermediate c.e. degrees, i.e. not low\( _n \), not high\( _n \) for any \( n \).
Superlow sets and $wtt$-noncuppable

A set $A$ is superlow, if $A' \leq_{tt} \emptyset'$. A set $B$ is superhigh, if $\emptyset'' \leq_{tt} H'$. There are superlow c.e. sets $A$ and $B$ such that $\emptyset' \leq_T A \oplus B$. (Bickford and Mills, 1982)

We cannot strengthen "Turing reduction" above as "$wtt$-reduction", as Bickford and Mills also proved.

If $A$ is superlow and $\emptyset'$ is $wtt$-reducible to $A \oplus W$, then $\emptyset'$ is $wtt$-reducible to $W$. This shows the existence of c.e. sets, low, but not superlow.
Superlow sets and \textit{wtt}-noncuppable

- A set $A$ is superlow, if $A' \leq_{tt} \emptyset'$.

- A set $B$ is superhigh, if $\emptyset'' \leq_{tt} H'$.
Superlow sets and \textit{wtt}-noncuppable

- A set $A$ is superlow, if $A' \leq_{tt} \emptyset'$.

- A set $B$ is superhigh, if $\emptyset'' \leq_{tt} H'$.

- There are superlow c.e. sets $A$ and $B$ such that $\emptyset' \leq_T A \oplus B$. (Bickford and Mills, 1982)

We cannot strengthen “Turing reduction” above as “\textit{wtt}-reduction”, as Bickford and Mills also proved.

If $A$ is superlow and $\emptyset'$ is \textit{wtt}-reducible to $A \oplus W$, then $\emptyset'$ is \textit{wtt}-reducible to $W$.

This shows the existence of c.e. sets, low, but not superlow.
Superlow sets and \textit{wtt}-noncuppable

- A set $A$ is superlow, if $A' \leq_{tt} \emptyset'$.

- A set $B$ is superhigh, if $\emptyset'' \leq_{tt} H'$.

- There are superlow c.e. sets $A$ and $B$ such that $\emptyset' \leq_T A \oplus B$. (Bickford and Mills, 1982)

- We cannot strengthen “Turing reduction” above as “\textit{wtt}-reduction”, as Bickford and Mills also proved
  - If $A$ is superlow and $\emptyset'$ is \textit{wtt}-reducible to $A \oplus W$, then $\emptyset'$ is \textit{wtt}-reducible to $W$. 
Superlow sets and \textit{wtt}-noncuppable

- A set $A$ is superlow, if $A' \leq_{tt} \emptyset'$.

- A set $B$ is superhigh, if $\emptyset'' \leq_{tt} H'$.

- There are superlow c.e. sets $A$ and $B$ such that $\emptyset' \leq_T A \oplus B$. (Bickford and Mills, 1982)

- We cannot strengthen “Turing reduction” above as “\textit{wtt}-reduction”, as Bickford and Mills also proved

  - If $A$ is superlow and $\emptyset'$ is \textit{wtt}-reducible to $A \oplus W$, then $\emptyset'$ is \textit{wtt}-reducible to $W$.

- This shows the existence of c.e. sets, low, but not superlow.
For $A \subseteq \mathbb{N}$, define

$$A^\dagger = \{x : \exists i < x[\varphi_i(x) \downarrow \land \Phi^A_x\varphi_i(x)(x) \downarrow]\}.$$ 

Obviously, $A^\dagger \leq_T A \oplus \emptyset'$, so if $A \geq_T \emptyset'$, then $A^\dagger \equiv_T A$. 

As indicated above, $A^\dagger \leq_T A \oplus \emptyset'$ is always true.
Bounded Jump Operator - a definition of Anderson and Csima

For $A \subseteq \mathbb{N}$, define

$$A^\dagger = \{x : \exists i < x[\varphi_i(x) \downarrow \& \Phi_x^{A^\dagger\varphi_i(x)}(x) \downarrow]\}.$$ 

Obviously, $A^\dagger \leq_T A \oplus \emptyset'$, so if $A \geq_T \emptyset'$, then $A^\dagger \equiv_T A$.

We can also have:

- $\emptyset^\dagger \equiv_1 \emptyset'$.
  So, for set $A$, $A \leq_{wtt} \emptyset^\dagger$ if and only if $A$ is $\omega$-c.e.
- $A \leq_1 A^\dagger$ and
  - $A^\dagger \not\leq_{wtt} A$.
  - $A^\dagger \leq_1 A'$.
- For some set $A$, $A^\dagger \not\leq_{wtt} A \oplus \emptyset'$.
  As indicated above, $A^\dagger \leq_T A \oplus \emptyset'$ is always true.
- For sets $A, B$ with $A \leq_{wtt} B$, $A^\dagger \leq_1 B^\dagger$.
  This shows that $\dagger$, as a jump, is well-defined.
An analogue of Shoenfield’s Jump Inversion

Theorem (Anderson and Csima):
For a set \( C \) with \( \emptyset^\dagger \leq_{\text{wtt}} C \leq_{\text{wtt}} \emptyset^{\dagger\dagger} \), there is a set \( B \leq_{\text{wtt}} \emptyset^\dagger \) such that \( C \equiv_{\text{wtt}} B^\dagger \).

1. All superlow sets are bounded-low.
2. Bounded-low sets can have high degree. (Anderson, Csima and Lange)
3. There is a superhigh bounded-low set. (G. Wu and H. Wu)
4. There is a low, but not superlow, bounded-low set. (G. Wu and H. Wu)
Theorem (Anderson and Csima):
For a set $C$ with $\emptyset^\dagger \leq_{wtt} C \leq_{wtt} \emptyset^{\dagger\dagger}$, there is a set $B \leq_{wtt} \emptyset^\dagger$ such that $C \equiv_{wtt} B^\dagger$.

A set $A$ is bounded-low if $A^\dagger \leq_{wtt} \emptyset^\dagger$, i.e. if $A^\dagger$ is $\omega$-c.e..

1. All superlow sets are bounded-low.

2. Bounded-low sets can have high degree. (Anderson, Csima and Lange)

3. There is a superhigh bounded-low set. (G. Wu and H. Wu)

4. There is a low, but not superlow, bounded-low set. (G. Wu and H. Wu)
The structure of r.e. \textit{wtt}-degrees

1. A splitting theorem of Ladner and Sasso, in contrast to Lachlan’s nonsplitting theorem

2. Why does it work for \textit{wtt}-degrees?

3. Distributivity, by Lachlan
Contiguous degrees

1. Definition and characterization of contiguous degrees

2. Contiguous degrees are low₂

3. Downey’s strongly contiguous degrees and noncuppability

4. Stob’s result of a contiguous degree as a top of minimal pair, and discontinuity

5. dre contiguous degree, and others (joint with Yamaleev)
No maximal dre \textit{wtt}-degrees

1. CHLLS's maximal dre Turing degrees

2. Maximal dre Turing degrees cannot be low, by ACL and DY.

3. There are no maximal dre \textit{wtt}-degrees (joint with Yamaleev)

4. Requirements and strategies
Thanks!