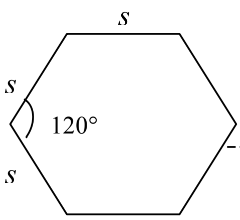
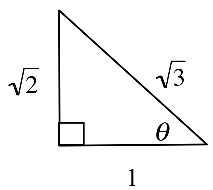
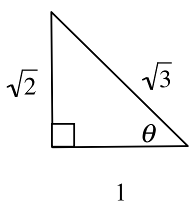
Assessment Schedule – 2008

Scholarship Mathematics with Calculus (93202)

Evidence Statement

For up to a maximum of 4 question parts a single minor error in that part may be accepted without loss of marks.

Question	Evidence	Code	Judgement
ONE (a)	 Using 6 equilateral triangles: Area $= 6\times\frac{1}{2}s^2\sin 60°$ $= 3s^2\sin 60°$	2	
	$= \frac{3\sqrt{3}}{2}s^2$ **OR** With two triangles and a rectangle: Area $= 2\times\frac{1}{2}s^2\sin 120° + 2s^2\sin 60°$ $= 3s^2\sin 60°$	1	
	$= \frac{3\sqrt{3}}{2}s^2$ Total area of the hexagonal stack $= 16\times\frac{3\sqrt{3}}{2}s^2$ $= 24\sqrt{3}s^2$	1	
ONE (b)	$A = 6hs + \frac{3}{2}s^2\left(\frac{-\cos\theta}{\sin\theta} + \frac{\sqrt{3}}{\sin\theta}\right) = 6hs + \frac{3}{2}s^2\left(-\cot\theta + \sqrt{3}\operatorname{cosec}\theta\right)$ So $\frac{dA}{d\theta} = \frac{3}{2}s^2\left(\operatorname{cosec}^2\theta - \sqrt{3}\operatorname{cosec}\theta\cot\theta\right)$ and for max/min	3	
	$\frac{3}{2}s^2\left(\operatorname{cosec}^2\theta - \sqrt{3}\operatorname{cosec}\theta\cot\theta\right) = 0$ $\operatorname{cosec}\theta\left(\operatorname{cosec}\theta - \sqrt{3}\cot\theta\right) = 0$ $\operatorname{cosec}^2\theta\left(1 - \sqrt{3}\cos\theta\right) = 0$  	1	
	But $\operatorname{cosec}^2\theta \neq 0$ so $1 - \sqrt{3}\cos\theta = 0$ $\cos\theta = \frac{1}{\sqrt{3}}$ **OR** $A = 6hs + \frac{3}{2}s^2\left(\frac{-\cos\theta}{\sin\theta} + \frac{\sqrt{3}}{\sin\theta}\right)$ $\frac{dA}{d\theta} = \frac{3}{2}s^2\left(\frac{\sin\theta(\sin\theta) - \left(-\cos\theta + \sqrt{3}\right)\cos\theta}{\sin^2\theta}\right)$	2 1	

		1	
	$= \frac{3}{2}s^2\left(\frac{\sin^2\theta + \cos^2\theta - \sqrt{3}\cos\theta}{\sin^2\theta}\right)$ $= \frac{3}{2}s^2\left(\frac{1 - \sqrt{3}\cos\theta}{\sin^2\theta}\right) = 0$ for max and min	2	
	$1 - \sqrt{3}\cos\theta = 0$ $\cos\theta = \frac{1}{\sqrt{3}}$ Hence, surface area is a minimum at $\theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$ $(= 0.9553 \text{ radians})$ $(= 54.7^\circ)$		
ONE (c)	$\frac{\mathrm{d}s}{\mathrm{d}t} = k\sin\theta$ $k\sin\frac{\pi}{4} = \sqrt{2}$ $k = 2$	3 1	Both needed
	$A = 6hs + \frac{3}{2}s^2\left(-\cot\theta + \sqrt{3}\operatorname{cosec}\theta\right)$ $\frac{\mathrm{dA}}{\mathrm{d}t} = 6h\frac{\mathrm{d}s}{\mathrm{d}t} + \frac{3}{2}2s\frac{\mathrm{d}s}{\mathrm{d}t}\left(-\cot\theta + \sqrt{3}\operatorname{cosec}\theta\right) + \frac{3}{2}s^2\left(\operatorname{cosec}^2\theta - \sqrt{3}\operatorname{cosec}\theta\cot\theta\right)\frac{\mathrm{d}\theta}{\mathrm{d}t}$ $= 2\sin\theta\left(6h + 3s\left(-\cot\theta + \sqrt{3}\operatorname{cosec}\theta\right)\right) + \frac{3}{2}s^2\operatorname{cosec}\theta\frac{\mathrm{d}\theta}{\mathrm{d}t}\left(\operatorname{cosec}\theta - \sqrt{3}\cot\theta\right)$	2	
	$= \frac{2\sqrt{2}}{\sqrt{3}}\left(6h + 3s\left(-\frac{1}{\sqrt{2}} + \sqrt{3}\frac{\sqrt{3}}{\sqrt{2}}\right)\right) + \frac{3}{2}s^2\sqrt{\frac{3}{2}}\frac{\mathrm{d}\theta}{\mathrm{d}t}\left(\sqrt{\frac{3}{2}} - \sqrt{3}\frac{1}{\sqrt{2}}\right)$ $= \frac{2\sqrt{2}}{\sqrt{3}}\left(6h + 3s\frac{2}{\sqrt{2}}\right)$ $= 4\left(\sqrt{6}h + \sqrt{3}s\right)$ (increasing, –sign not needed) 		Or equivalent

Assessment Schedule – 2013

Scholarship Calculus (93202)

Evidence Statement

Question One

(a) Solving $\frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{\varphi}\frac{2\mathrm{e}^{-2x} - \mathrm{e}^{-x}}{2\sqrt{\mathrm{e}^{-x} - \mathrm{e}^{-2x}}} = \sqrt{\varphi}\frac{2\mathrm{e}^{-2x} - \mathrm{e}^{-x}}{2y} = 0$, we find

$$\begin{aligned} 2\mathrm{e}^{-2x} - \mathrm{e}^{-x} &= 0 \\ \ln 2 - 2x &= -x \\ x &= \ln 2 \end{aligned}$$

The drop is widest at $x = \ln 2 \approx 0.6931$, and then $y = \sqrt{\varphi}\sqrt{\mathrm{e}^{-\ln 2} - \mathrm{e}^{-2\ln 2}} = \sqrt{\varphi}\sqrt{\frac{1}{2} - \frac{1}{4}} = \frac{\sqrt{\varphi}}{2} \approx 0.6360$. It is widest $\ln 2$ cm from the rounded end **C**, and is exactly $\sqrt{\varphi}$ cm wide there.

(b) Now we need $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$, so $\mathrm{e}^{2x} - 6\mathrm{e}^{x} + 4 = 0$.

-Solving as a quadratic in e^x we find $\mathrm{e}^x = \frac{-(-6) \pm \sqrt{20}}{2} = 3 \pm \sqrt{5}$, so $x = \ln\left(3 \pm \sqrt{5}\right)$.

Since $0 < 3 - \sqrt{5} < 1$, $x = \ln\left(3 - \sqrt{5}\right) < 0$ and so this point is not valid, and the only turning point is at $x = \ln\left(3 + \sqrt{5}\right)$

(c)

$$\begin{aligned} \int_0^{\ln p} \pi y^2 \,\mathrm{d}x &= \pi\varphi \int_0^{\ln p} \left(\mathrm{e}^{-x} - \mathrm{e}^{-2x}\right)\mathrm{d}x \\ &= \pi\varphi\left[\tfrac{1}{2}\mathrm{e}^{-2x} - \mathrm{e}^{-x}\right]_0^{\ln p} \\ &= \pi\varphi\left(\tfrac{1}{2} + \left(\tfrac{1}{2}\mathrm{e}^{-2\ln p} - \mathrm{e}^{-\ln p}\right)\right) \\ &= \pi\varphi\left(\tfrac{1}{2} + \tfrac{1}{2}\tfrac{1}{p^2} + \tfrac{1}{p}\right) \\ &= \pi\varphi\tfrac{p^2 - 2p + 1}{2p^2} \\ &= \pi\varphi\tfrac{(p-1)^2}{2p^2} \\ &= \tfrac{1}{2}\pi\varphi\left(\frac{p-1}{p}\right)^2 \end{aligned}$$

1(a) Expression for $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ [1st mark],
widest at $x = \ln 2$ [2nd mark],
width is $\sqrt{\varphi}$ [3rd mark].

1(b) Recognise quadratic $a^2 - 6a + 4 = 0$ and solve [1st mark],
find $x = \ln\left(3 + \sqrt{5}\right)$ [2nd mark].

1(c) Definite integral $= \pi\varphi\left[\tfrac{1}{2}e^{-2x} - e^{-x}\right]_0^{\ln p}$ [1st mark],
show required form $= \tfrac{1}{2}\pi\varphi\left(\tfrac{p-1}{p}\right)^2$ [2nd mark],
explain upper limit [3rd mark].

In all questions, minor error ignored once if one single character is incorrect, inserted or omitted.
Note that the 2-mark question is not always part (a).

Since $p - 1 < p$ and $\ln p \geq 0$ for the model, we can see that $0 \leq \frac{p-1}{p} < 1$ and so $0 \leq V < \frac{1}{2}\pi\varphi$.

(As *p* gets larger, the volume approaches $\frac{1}{2}\pi\varphi \approx 2.54\ \mathrm{cm}^3$. In the diagram, the drop has 99.99% of the maximal volume in the model.)

Marks in each question part are independent, with follow through marks.

Question Two

(a) The two functions are orthogonal if $\theta = \frac{\pi}{2}$, so $0 = \langle f, g \rangle_0^1$.

$$\langle f, g \rangle_0^1 = \int_0^1 (kx+1)(x+k)\,dx$$
$$= \int_0^1 \left(kx^2 + (k^2+1)x + k\right)dx$$
$$= \left[\frac{kx^3}{3} + \frac{(k^2+1)x^2}{2} + kx\right]_0^1$$
$$= \frac{k}{3} + \frac{(k^2+1)}{2} + k = 0$$
$$k^2 + \frac{8}{3}k + 1 = 0$$
$$2k = -\frac{8}{3} \pm \sqrt{\frac{64}{9} - 4} = -\frac{8}{3} \pm \sqrt{\frac{28}{9}} = -\frac{8}{3} \pm \frac{2}{3}\sqrt{7}$$
$$k = -\frac{4}{3} \pm \frac{1}{3}\sqrt{7} = \frac{-4 \pm \sqrt{7}}{3}$$

2(a) Definite integral [1st mark],
values $k = \frac{-4\pm\sqrt{7}}{3}$ [2nd mark].
2(b) Find at least two of $\|f\|, \|g\|, \langle f, g \rangle_0^1$ [1st mark],
find $\cos\theta = \frac{7/2}{\sqrt{7}\sqrt{7}}$ [2nd mark],
and $\theta = \frac{\pi}{3}$ [3rd mark].
2(c) Either trig identity used, or first step of integration by parts [1st mark],
show that $\langle \ldots \rangle_0^1 = 0$ when $m \neq n$ [2nd mark],
test what happens when $n = m$ [3rd mark].

(b) The angle requires finding three inner products, since $\langle f, g \rangle_0^1 \neq 0$.

$$\langle f, g \rangle_0^1 = \int_0^1 (3x-4)(9x-5)\,dx = \int_0^1 \left(27x^2 - 51x + 20\right)dx = \left[9x^3 - \frac{51}{2}x^2 + 20x\right]_0^1 = 9 - \frac{51}{2} + 20 = \frac{7}{2}$$
$$\langle f, f \rangle_0^1 = \int_0^1 (3x-4)(3x-4)\,dx = \int_0^1 \left(9x^2 - 24x + 16\right)dx = \left[3x^3 - 12x^2 + 16x\right]_0^1 = 3 - 12 + 16 = 7$$
$$\langle g, g \rangle_0^1 = \int_0^1 (9x-5)(9x-5)\,dx = \int_0^1 \left(81x^2 - 90x + 25\right)dx = \left[27x^3 - 45x^2 + 25x\right]_0^1 = 27 - 45 + 25 = 7$$
$$\cos\theta = \frac{\langle f, g \rangle_0^1}{\sqrt{\langle f, f \rangle_0^1 \cdot \langle g, g \rangle_0^1}} = \frac{7}{2} \div 7 = \frac{1}{2} \qquad \text{so } \theta = \cos^{-1}\frac{1}{2} = \frac{\pi}{3}$$

(c) $$\int_0^{2\pi} \sin(mx)\sin(nx)\,dx = \frac{1}{2}\int_0^{2\pi} \left(\cos((m-n)x) - \cos((m+n)x)\right)dx$$
$$= \frac{1}{2}\left[\frac{\sin((m-n)x)}{m-n} - \frac{\sin((m+n)x)}{m+n}\right]_0^{2\pi} = 0 \text{ if } m \neq n$$

However, if $n = m$
$$\int_0^{2\pi} \sin(mx)\sin(nx)\,dx = \frac{1}{2}\int_0^{2\pi} \left(1 - \cos(2nx)\right)dx = \frac{1}{2}\left[x - \frac{\sin(2nx)}{2n}\right]_0^{2\pi} = \pi - \frac{\sin(4n\pi)}{4n} = \pi \neq 0.$$

So $\sin(nx)$ and $\sin(mx)$ are othogonal if $m \neq n$.

(see also integration by parts in appendix)

Question Three

(a) (i) A polynomial $p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n$ is:
even if and only if the odd coefficients are all zero, and
odd if and only if the even coefficients are all zero.
neither if there are both even and odd coefficients which are non-zero.

Candidates might note that $p(x) = 0$ is the unique polynomial which is both even and odd.

[It is *insufficient* to say that $p(x) = ax^{2n}$ is even, and $p(x) = ax^{2n+1}$ is odd, unless also noting that the sum of odd polynomials is odd, and the sum of even polynomials is even.]

(ii) We are given that g is an even function, so $g(-x) = g(x)$ for all x.
We use the facts that $g(-x+h) = g(x-h)$ and $g(-x) = g(x)$.

$$\begin{aligned} g'(-x) &= \lim_{h\to 0} \frac{g(-x+h)-g(-x)}{h} \\ &= \lim_{h\to 0} \frac{g(x-h)-g(x)}{h} \\ &= \lim_{k\to 0} \frac{g(x+k)-g(x)}{-k} \quad \text{where } k=-h \\ &= -\lim_{k\to 0} \frac{g(x+k)-g(x)}{k} \\ &= -g'(x) \end{aligned}$$

Since $g'(-x) = -g'(x)$ we have shown that $\frac{\mathrm{d}g}{\mathrm{d}x}$ is an odd function.

3(a)(i) Recognise building block functions (powers) as odd and even [1st mark],
full description of odd, even and neither [2nd mark].
3(a)(ii) Write expression for $g'(-x)$ [1st mark],
use $g(-x+h) = g(x-h)$ [2nd mark],
full proof [3rd mark].
3(b) Find $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ [1st mark],
get form $\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = Ay + Be^{-x}\cos kx$ [2nd mark],
find $C = 8$ [3rd mark].

(b) We find the third derivative, then look to the coefficients of the terms.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\mathrm{e}^{-x}\sin(kx) + k\mathrm{e}^{-x}\cos(kx)$$

$$\begin{aligned} \frac{\mathrm{d}^2y}{\mathrm{d}x^2} &= \mathrm{e}^{-x}\sin(kx) - k\mathrm{e}^{-x}\cos(kx) - k\mathrm{e}^{-x}\cos(kx) - k^2\mathrm{e}^{-x}\sin(kx) \\ &= (1-k^2)\mathrm{e}^{-x}\sin(kx) - 2k\mathrm{e}^{-x}\cos(kx) \end{aligned}$$

$$\begin{aligned} \frac{\mathrm{d}^3y}{\mathrm{d}x^3} &= (k^2-1)\mathrm{e}^{-x}\sin(kx) + (k-k^3)\mathrm{e}^{-x}\cos(kx) + 2k\mathrm{e}^{-x}\cos(kx) + 2k^2\mathrm{e}^{-x}\sin(kx) \\ &= (3k^2-1)\mathrm{e}^{-x}\sin(kx) + (3k-k^3)\mathrm{e}^{-x}\cos(kx) \\ &= (3k^2-1)y + k(3-k^2)\mathrm{e}^{-x}\cos(kx) \end{aligned}$$

To have $\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = Cy$ we must have $k(3-k^2) = 0$ so $k = \pm\sqrt{3}$ and then $C = 3k^2 - 1 = 8$.

Question Four

(a) For each $2 \le n \le 9$ we list the solutions of $z^n = z$; these are the solutions of $z^{n-1} = 1$; roots of unity.

We need to be careful not to count any roots twice; this is easiest when writing the solutions in the form $z = \text{cis}\left(\frac{a}{n-1}\pi\right)$.

$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$	$n=8$	$n=9$
$z=\text{cis}\,0\pi$	$z=\text{cis}\,\pi$	$z=\text{cis}\frac{2}{3}\pi$ $z=\text{cis}\frac{4}{3}\pi$	$z=\text{cis}\frac{1}{2}\pi$ $z=\text{cis}\frac{3}{2}\pi$	$z=\text{cis}\frac{2}{5}\pi$ $z=\text{cis}\frac{4}{5}\pi$ $z=\text{cis}\frac{6}{5}\pi$ $z=\text{cis}\frac{8}{5}\pi$	$z=\text{cis}\frac{1}{3}\pi$ $z=\text{cis}\frac{5}{3}\pi$	$z=\text{cis}\frac{2}{7}\pi$ $z=\text{cis}\frac{4}{7}\pi$ $z=\text{cis}\frac{6}{7}\pi$ $z=\text{cis}\frac{8}{7}\pi$ $z=\text{cis}\frac{10}{7}\pi$ $z=\text{cis}\frac{12}{7}\pi$	$z=\text{cis}\frac{1}{4}\pi$ $z=\text{cis}\frac{3}{4}\pi$ $z=\text{cis}\frac{5}{4}\pi$ $z=\text{cis}\frac{7}{4}\pi$
1	1	2	2	4	2	6	4

There are 22 solutions in the form $z = \text{cis}\left(\frac{a}{n-1}\pi\right)$, and also the solution $z = 0$.

There are **23 solutions in total**.

(b)(i)

We rearrange to collect Δv terms together.

$$\left(\frac{m_0}{m_1}\right)^{\frac{2u}{c}} = \frac{1+\frac{\Delta v}{c}}{1-\frac{\Delta v}{c}}$$

$$\left(\frac{m_0}{m_1}\right)^{\frac{2u}{c}}\left(1-\frac{\Delta v}{c}\right) = 1+\frac{\Delta v}{c}$$

$$\left(\frac{m_0}{m_1}\right)^{\frac{2u}{c}} - \left(\frac{m_0}{m_1}\right)^{\frac{2u}{c}}\frac{\Delta v}{c} = 1+\frac{\Delta v}{c}$$

$$\left(\frac{m_0}{m_1}\right)^{\frac{2u}{c}} - 1 = \frac{\Delta v}{c}\left(\left(\frac{m_0}{m_1}\right)^{\frac{2u}{c}} + 1\right)$$

$$\Delta v = c\frac{\left(\frac{m_0}{m_1}\right)^{\frac{2u}{c}} - 1}{\left(\frac{m_0}{m_1}\right)^{\frac{2u}{c}} + 1}$$

$$= c\frac{\exp\left(2\ln\left(\frac{m_0}{m_1}\right)\frac{u}{c}\right)-1}{\exp\left(2\ln\left(\frac{m_0}{m_1}\right)\frac{u}{c}\right)+1}$$

$$= c\tanh\left(\frac{u}{c}\ln\left(\frac{m_0}{m_1}\right)\right)$$

Also possible to work this backwards.

4(a) Understanding of roots of unity (could be shown in a diagram) [1st mark],
find all non-zero roots (repetition allowed) [2nd mark],
23 solutions in total (allow MEI for 22) [3rd mark].
4(b)(i) Multiply out $\left(1-\frac{\Delta v}{c}\right)$ [1st mark],
get Δv as subject, any form [2nd mark],
required form [3rd mark].
OR an equivalent form in the opposite direction, with substituting $\tanh$ correctly as [1st mark].
4(b)(ii) Differentiate both sides, or integrate with partial fractions [1st mark],
into required form [2nd mark].

(b)(ii)

We differentiate the given equation and show it satisfies the differential equation.

$$\frac{\mathrm{d}}{\mathrm{d}v}\left(\ln M\right) = -\frac{c}{2u}\frac{\mathrm{d}}{\mathrm{d}v}\left(\ln\left(\frac{1+v/c}{1-v/c}\right)\right)$$

$$\frac{1}{M}\frac{\mathrm{d}M}{\mathrm{d}v} = -\frac{c}{2u}\times\frac{\left(1-v/c\right)}{\left(1+v/c\right)}\times\frac{\frac{1}{c}\left(1-v/c\right)-\frac{-1}{c}\left(1+v/c\right)}{\left(1-v/c\right)^2}$$

$$\frac{\mathrm{d}M}{\mathrm{d}v} = \frac{-Mc}{2u}\frac{\frac{2}{c}}{\left(1+v/c\right)\left(1-v/c\right)}$$

$$= \frac{-M}{u\left(1-v^2/c^2\right)}$$

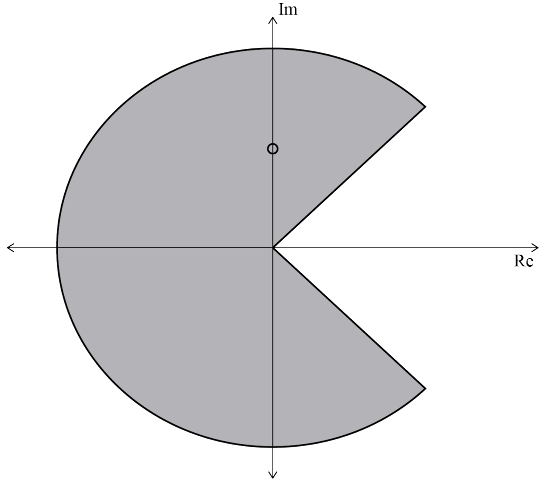
QUESTION THREE SOLUTIONS

(a) (i) $\frac{d}{dx}(x\cos x) = \cos x - x\sin x$

Hence by observation, $\int x\sin x\,dx = \sin x - x\cos x + c$

(ii) $\int_0^{n\pi} x\sin x\,dx = \left[\sin x - x\cos x\right]_0^{n\pi} = \sin(n\pi) - n\pi\cos(n\pi) = -n\pi\cos(n\pi) = (-1)^{n+1}n\pi$

(b) Diagram should also be labelled, or otherwise described, to show the radius is also 1, and the 'eye' is at $0 + 0.512\text{i}$. The 'mouth' is open to an angle of $\frac{\pi}{2}$, with one edge at an angle of $\frac{\pi}{4}$



THREE(a)(i)

1	$\frac{dy}{dx} = \cos x - x\sin x$	
1	hence $\int x\sin x\,dx = \sin x - x\cos x$	by observation from the result
1	$+c$	the constant must be there

THREE(a)(ii)

1	$= \sin(n\pi) - n\pi\cos(n\pi)$	find definite integral
1	$\sin(n\pi) = 0$	note this
to 3	$(-1)^{n+1}n\pi = \begin{cases} n\pi \text{ if } n \text{ is odd} \\ -n\pi \text{ if } n \text{ is even} \end{cases}$	either form, with specific cases: $\pm n\pi$ is not sufficient

THREE(b)

1	pacman shape: sector between 180 and 360 degrees	mouth open to the right
1	correct z^3 for the eye	close to 0.5i, on imaginary axis
1	pacman mouth open at 90 degrees, 45 degrees above and below	either labelled with angles in diagram, or angles calculated in working elsewhere

QUESTION FOUR SOLUTIONS

(a)

$f(x)=\log_m x+\log_x m=\frac{\ln x}{\ln m}+\frac{\ln m}{\ln x}$	using identities for logarithms
$\frac{df}{dx}=\frac{1}{\ln m}\frac{1}{x}+\ln m\frac{1}{x}\frac{-1}{(\ln x)^2}$	using the chain (or quotient) rule
$\frac{df}{dx}=0$	setting derivative to zero to find critical point
$\frac{1}{\ln m}\frac{1}{x}=\frac{1}{x}\frac{\ln m}{(\ln x)^2}$	rearranging to get in terms of x
$(\ln x)^2=(\ln m)^2$	note that $x\neq 0$
$\ln x=\pm\ln m$	positive and negative solutions are possible,
$x=m$ or $x=\frac{1}{m}$	but we rule out the second as $m,x>1$

Minimum value is $\log_m m+\log_m m=2$

FOUR(a)		
1	find $\frac{df}{dx}=\frac{1}{x\ln m}-\frac{\ln m}{x(\ln x)^2}$	any form
to 2	minimum value is at $f(m)=2$	**evidence** required (minimum test not required)
+1	“clearly explain the steps of your working” explains correct and valid steps	need not be as complete as shown, but must explain at least TWO key decisions made

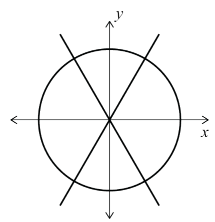
(b) Substituting, we find $y^2-q^2x^2$ is a factor:

$$y^4+(1-q^2)x^2y^2-q^2x^4+q^2x^2-y^2=0$$
$$q^4x^4+(1-q^2)q^2x^4-q^2x^4+q^2x^2-q^2x^2=0$$
$$q^4x^4+q^2x^4-q^4x^4-q^2x^4+q^2x^2-q^2x^2=0$$
$$0=0$$

So we can factorise:

$$y^4+(1-q^2)x^2y^2-q^2x^4+q^2x^2-y^2=0$$
$$(y^2-q^2x^2)(y^2+x^2-1)=0$$
$$(y+qx)(y-qx)(y^2+x^2-1)=0$$

This gives the lines $y=\pm qx$ and the unit circle $x^2+y^2=1$.



FOUR(b)		
1	both $y=\pm qx$ as equations of lines	need not be drawn, could be labelled in diagram
1	factorise to find other factor x^2+y^2-1	need not recognise as a circle
1	diagram with circle and lines crossing at centre	lines of any slopes $\pm q$ with reflection symmetry. Axes not required. Labelling radius $r=1$ not required.

(c)

$$\tan 4x\left(\tan^2 x - 2\tan x - 1\right)\left(\tan^2 x + 2\tan x - 1\right)$$

$$= \frac{2\tan 2x}{1-\tan^2 2x}\left(\tan^2 x - 2\tan x - 1\right)\left(\tan^2 x + 2\tan x - 1\right)$$

$$= \frac{2\tan 2x}{1-\left(\frac{2\tan x}{1-\tan^2 x}\right)^2}\left(\tan^2 x - 2\tan x - 1\right)\left(\tan^2 x + 2\tan x - 1\right)$$

$$= \frac{2\tan 2x\left(1-\tan^2 x\right)^2}{\left(1-\tan^2 x\right)^2 - 4\tan^2 x}\left(\tan^2 x - 2\tan x - 1\right)\left(\tan^2 x + 2\tan x - 1\right)$$

$$= \frac{2\tan 2x\left(1-\tan^2 x\right)^2}{\left(1-\tan^2 x - 2\tan x\right)\left(1-\tan^2 x + 2\tan x\right)}\left(\tan^2 x - 2\tan x - 1\right)\left(\tan^2 x + 2\tan x - 1\right)$$

$$= 2\tan 2x\left(1-\tan^2 x\right)^2 = 2\tan 2x\left(\tan^2 x - 1\right)^2 = 2\tan 2x\left(\tan x - 1\right)^2\left(\tan x + 1\right)^2$$

OR

Using $\tan 4x = \frac{2\tan 2x}{1-\tan^2 2x} \Rightarrow 2\tan 2x = \tan 4x\left(1-\tan^2 2x\right)$

$$2\tan 2x\left(\tan x - 1\right)^2\left(\tan x + 1\right)^2 = \tan 4x\left(1-\tan^2 2x\right)\left(\tan x - 1\right)^2\left(\tan x + 1\right)^2$$

$$= \tan 4x\left(1-\tan^2 2x\right)\left(\tan^2 x - 1\right)^2$$

$$= \tan 4x\left(1-\left(\frac{2\tan x}{1-\tan^2 x}\right)^2\right)\left(\tan^2 x - 1\right)^2$$

$$= \tan 4x\left(\frac{\left(1-\tan^2 x\right)^2 - 4\tan^2 x}{\left(1-\tan^2 x\right)^2}\right)\left(\tan^2 x - 1\right)^2$$

$$= \tan 4x\left(\left(1-\tan^2 x\right)^2 - 4\tan^2 x\right)$$

$$= \tan 4x\left(1-\tan^2 x - 2\tan x\right)\left(1-\tan^2 x + 2\tan x\right)$$

$$= \tan 4x\left(\tan^2 x + 2\tan x - 1\right)\left(\tan^2 x - 2\tan x - 1\right)$$

FOUR(c)

Because there are different approaches possible, the marks are for two skills and then for completing the whole identity.

1	correct use of double angle formula for $\tan(4x)$	note: not if used on the LHS, this is unhelpful
1	RHS expanded to contain at most one $\tan 2x$ term	
to 3	full proof of identity given	any valid proof, regardless of approach, gets full marks

Converting to $\sin(x)$ and $\cos(x)$ is unlikely to lead to fruition.

Also note that that the expressions are equivalent to $\frac{8\tan^2 x}{\tan 2x}$

QUESTION TWO SOLUTION

(a) We aim to find $\left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{x=3}$ and we see that $\frac{\mathrm{d}V}{\mathrm{d}t} = 0.015 \times 25\pi = 0.375\pi$. Also $\frac{\mathrm{d}V}{\mathrm{d}x} = -25\pi + \pi x^2$.

$$\frac{dx}{dt} = \frac{dx}{dV}\frac{dV}{dt} = \frac{dV}{dt} \div \frac{dV}{dx} = \frac{0.375\pi}{(x^2-25)\pi} = \frac{0.375}{x^2-25}.$$

$\left.\frac{\mathrm{d}x}{\mathrm{d}t}\right|_{x=3} = \frac{0.375}{9-25} = -0.0234375$ metres per hour. The water is rising at (approximately) **23.4 mm per hour**.

(b)(i) Differentiating:

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}\left(A\left(1-\sqrt{x}\right)^{3/2}\left(2+3\sqrt{x}\right)+C\right) &= A\left(1-\sqrt{x}\right)^{3/2}\frac{3}{2\sqrt{x}} - \frac{3}{4\sqrt{x}}A\left(1-\sqrt{x}\right)^{1/2}\left(2+3\sqrt{x}\right) \\
&= \frac{3}{4\sqrt{x}}A\left(1-\sqrt{x}\right)^{1/2}\left[2\left(1-\sqrt{x}\right)-\left(2+3\sqrt{x}\right)\right] \\
&= \frac{3}{4\sqrt{x}}A\left(1-\sqrt{x}\right)^{1/2}\left[2-2\sqrt{x}-2-3\sqrt{x}\right] \\
&= \frac{3}{4\sqrt{x}}A\left(1-\sqrt{x}\right)^{1/2}\left(-5\sqrt{x}\right) \\
&= -\frac{15}{4}A\sqrt{1-\sqrt{x}}
\end{aligned}$$

and so $A = -\frac{4}{15}$, and the Fundamental Theorem of Calculus gives the integral as required.

(b)(ii) The area under the curve $y = g(x)$ is $\int_0^1 g(x)\mathrm{d}x = \left[-\frac{4}{15}\left(1-\sqrt{x}\right)^{3/2}\left(2+3\sqrt{x}\right)\right]_0^1 = \frac{8}{15}$.

The area beneath the dotted curve is the same as the area between $y = g(x)$ and $y = 1$; that is, $1 - \frac{8}{15} = \frac{7}{15}$.

So the area between the curves is $\frac{1}{15}$.