Unsolvable problems, the Continuum Hypothesis, and the nature of infinity

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The power set

Suppose $X$ is a set. The powerset of $X$ is the set

$$\mathcal{P}(X) = \{ Y \mid Y \text{ is a subset of } X \}.$$ 

Cumulative Hierarchy of Sets

The universe $V$ of sets is generated by defining $V_\alpha$ by induction on the ordinal $\alpha$:

1. $V_0 = \emptyset$,
2. $V_{\alpha+1} = \mathcal{P}(V_\alpha)$,
3. if $\alpha$ is a limit ordinal then $V_\alpha = \bigcup_{\beta < \alpha} V_\beta$.

▶ If $X$ is a set then $X \in V_\alpha$ for some ordinal $\alpha$. 

$V$: The Universe of Sets
\( V_0 = \emptyset, \ V_1 = \{\emptyset\}, \ V_2 = \{\emptyset, \{\emptyset\}\}. \)

- These are just the ordinals: 0, 1, and 2.

- \( V_3 \) has 4 elements.
  - This is not the ordinal 3 (in fact, it is not an ordinal).

- \( V_4 \) has 16 elements.

- \( V_5 \) has 65,536 elements.

- \( V_{1000} \) has a lot of elements.

\[ V_\omega \text{ is infinite, it is the set of all (hereditarily) finite sets.} \]

\[ \text{The conception of } V_\omega \text{ is} \mathbf{mathematically identical} \text{ to the conception of the structure} \ (\mathbb{N}, +, \cdot). \]
Beyond the basic axioms: large cardinal axioms

The axioms

- The ZFC axioms of Set Theory specify the basic axioms for $V$.
- These axioms are naturally augmented by additional principles which assert the existence of “very large” infinite sets.
  - These additional principles are called large cardinal axioms.

- There is a proper class of measurable cardinals.
- There is a proper class of strong cardinals.
- There is a proper class of Woodin cardinals.
- There is a proper class of superstrong cardinals.
- There is a proper class of supercompact cardinals.
- There is a proper class of extendible cardinals.
- There is a proper class of $\omega$-huge cardinals.
Cardinality: measuring the size of sets

**Definition:** when two sets have the same size

Two sets, $X$ and $Y$, have the same **cardinality** if there is a matching of the elements of $X$ with the elements of $Y$.

**Formally:** \(|X| = |Y|\) if there is a bijection $f : X \rightarrow Y$

- \(|\{0, 1, 2, \ldots, k, \ldots\}| = |\{1, 2, 3, \ldots, k, \ldots\}|$.
- \(|\mathbb{N}| = |\mathbb{Z}| = |\mathbb{Q}|$.
- \(|\mathbb{R}| = |\mathbb{R}^2|$ (not obvious at all!).

**Assuming the ZFC axioms:**

**Theorem (Cantor)**

For every set $X$ there is an ordinal $\alpha$ such that \(|X| = |\alpha|\).
The Continuum Hypothesis: CH

**Theorem (Cantor)**

The set $\mathbb{N}$ of all natural numbers and the set $\mathbb{R}$ of all real numbers do not have the same cardinality.

**The Continuum Hypothesis**

Suppose $A \subseteq \mathbb{R}$ is infinite. Then either:

1. $A$ and $\mathbb{N}$ have the same cardinality, or
2. $A$ and $\mathbb{R}$ have the same cardinality.

- This is Cantor’s Continuum Hypothesis.
- The Continuum Hypothesis holds for the simple infinite sets and it holds for many not-so-simple infinite sets.
Many tried to solve the problem of CH and failed.

In 1940, Gödel showed that it is consistent with the axioms of Set Theory that the Continuum Hypothesis be true.

▶ One cannot refute the Continuum Hypothesis.

In 1963, on July 4th, Cohen announced in a lecture at Berkeley that it is consistent with the axioms of Set Theory that the Continuum Hypothesis be false.

▶ One cannot prove the Continuum Hypothesis.
Cohen’s method

If $M$ is a universe of Set Theory then $M$ contains “blueprints” for virtual universes $N$ which extend $M$. These blueprints can be constructed and analyzed from within $M$.

- If $M$ is countable then every blueprint constructed within $M$ can be realized as genuine extension of $M$.

- Cohen proved that every universe $M$ contains a blueprint for an extension in which the Continuum Hypothesis is false.

- Cohen’s method also shows that every universe $M$ contains a blueprint for an extension in which the Continuum Hypothesis is true.

- (Levy-Solovay) These extensions preserve large cardinal axioms if these axioms hold for proper class of cardinals.
  - So if large cardinal axioms can help
    - it can only be in some indirect way.
The extent of Cohen’s method: It is not just about CH

A challenging time for the conception of $V$

- Cohen’s method of forcing has been vastly developed.
- Many questions have been showed to be unsolvable.

- A serious challenge to the very conception of Mathematical Infinity.
  - The Continuum Hypothesis is a statement about just $V_{\omega+2}$. 
We have a problem
Question

Is there a resolution?

- Perhaps one should begin by trying to understand CH

A natural conjecture

One can understand CH by looking at special cases.

- But which special cases?
  - Does this even make sense?
Seeking special cases to study for the problem of the Continuum Hypothesis

**Logical definability within a set $X$**

Suppose that $X$ is a set. A subset $Y \subseteq X$ is logically definable in $X$ from parameters if there is a formal property $\varphi(x_0, \ldots, x_n)$ and elements $a_1, \ldots, a_n$ of $X$ such that

- $Y$ is the set of all $a \in X$ such that $\varphi[a, a_1, \ldots, a_n]$ holds in $X$.

**The definable power set**

For each set $X$, $\mathcal{P}_{\text{Def}}(X)$ denotes the set of all $Y \subseteq X$ such that $Y$ is logically definable in the structure $(X, \in)$ from parameters in $X$.

- $\mathcal{P}_{\text{Def}}(X)$ is the collection of all “specifiable” subsets of $X$ versus $\mathcal{P}(X)$ which is the collection of all subsets of $X$. 
The projective sets

Definition

A set $A \subseteq \mathbb{R}$ is *projective* if it can be generated from the open subsets of $\mathbb{R}$ in finitely many steps of taking complements and images by continuous functions, $f : \mathbb{R} \to \mathbb{R}$.

- The study of the projective sets was a major focus of the Polish School of Mathematics in the early 1900s.
- $\mathbb{R} \subset V_{\omega+1}$ and so $\mathcal{P}(\mathbb{R}) \subset \mathcal{P}(V_{\omega+1}) = V_{\omega+2}$.

The collection of projective sets is exactly:

$$\mathcal{P}(\mathbb{R}) \cap \mathcal{P}_{\text{Def}}(V_{\omega+1})$$
The Continuum Hypothesis and the Projective Sets

The projective Continuum Hypothesis

Suppose $A \subseteq \mathbb{R}$ is an infinite projective set. Then either:

1. $A$ and $\mathbb{N}$ have the same cardinality, or
2. $A$ and $\mathbb{R}$ have the same cardinality.

- There were many attempts in the early 1900s to prove the projective Continuum Hypothesis with success for the simplest instances.
- However, by 1925 it too began to look like a hopeless problem.
It was a hopeless problem

The actual constructions of Gödel and Cohen show that the projective Continuum Hypothesis is also formally unsolvable.

- This explains why the problem of the projective Continuum Hypothesis was so difficult.
- But the intuition that the problem has an answer was correct.

Theorem

Suppose there are infinitely many Woodin cardinals. Then the strong projective Continuum Hypothesis is true:

- Every uncountable projective set contains an uncountable closed set.

- Maybe the problem of the Continuum Hypothesis also has an answer
  - and maybe the key is the definable powerset $\mathcal{P}_{\text{Def}}(X)$. 
The effective cumulative hierarchy: \( L \)

**Cumulative Hierarchy of Sets**

The cumulative hierarchy is defined by induction on \( \alpha \) as follows.

1. \( V_0 = \emptyset \).
2. \( V_{\alpha+1} = \mathcal{P}(V_\alpha) \).
3. if \( \alpha \) is a limit ordinal then \( V_\alpha = \bigcup_{\beta < \alpha} V_\beta \).

\( V \) is the class of all sets \( X \) such that \( X \in V_\alpha \) for some \( \alpha \).

**Gödel’s constructible universe, \( L \)**

Define \( L_\alpha \) by induction on \( \alpha \) as follows.

1. \( L_0 = \emptyset \).
2. \( L_{\alpha+1} = \mathcal{P}_{\text{Def}}(L_\alpha) \).
3. if \( \alpha \) is a limit ordinal then \( L_\alpha = \bigcup \{ L_\beta \mid \beta < \alpha \} \).

\( L \) is the class of all sets \( X \) such that \( X \in L_\alpha \) for some \( \alpha \).
The axiom: \( V = L \)

Suppose \( X \) is a set. Then \( X \in L \).

Theorem (Gödel:1940)

Assume \( V = L \). Then the Continuum Hypothesis holds.

- Suppose there is a Cohen-blueprint for \( V = L \). Then:
  - the axiom \( V = L \) must hold and the blueprint is trivial.

Claim

Adopting the axiom \( V = L \) completely negates the ramifications of Cohen’s method.

- Could this be the resolution?
The axiom $V = L$ and large cardinals

Theorem (Scott:1961)

Assume $V = L$. Then there are no measurable cardinals.

- Then there are no (genuine) large cardinals.

- Assume $V = L$. Then there are no Woodin cardinals.

Clearly:

The axiom $V = L$ is false.

- We need to generalize $L$.
  - But how?
Towards generalizing the projective sets
More about the projective sets

Definition

A set $A \subseteq \mathbb{R}^n$ is **projective** if it can be generated from the open subsets of $\mathbb{R}^n$ in finitely many steps of taking complements and images by continuous functions,

$$f : \mathbb{R}^n \rightarrow \mathbb{R}^n.$$ 

Definition

Suppose that $A \subseteq \mathbb{R} \times \mathbb{R}$. A function $f$ **uniformizes** $A$ if for all $x \in \mathbb{R}$:

- if there exists $y \in \mathbb{R}$ such that $(x, y) \in A$ then $(x, f(x)) \in A$. 

Two questions of Luzin

1. Suppose $A \subseteq \mathbb{R} \times \mathbb{R}$ is projective. Can $A$ be uniformized by a projective function?

2. Suppose $A \subseteq \mathbb{R}$ is projective. Is $A$ Lebesgue measurable and does $A$ have the property of Baire?

Luzin’s questions are questions about $\langle \mathcal{P}(\mathbb{N}), \mathbb{N}, \cdot, +, \in \rangle$

Luzin conjectured in 1925 that “we will never know the answer to the measure question for the projective sets”.

Both questions are unsolvable on the basis of the ZFC axioms
Determinacy and the answers to Luzin’s questions

Suppose $A \subseteq \mathbb{R}$. There is an associated infinite game involving two players.

- The players alternate choosing $\epsilon_i \in \{0, 1\}$.
- After infinitely many moves an infinite binary sequence $\langle \epsilon_i : i \in \mathbb{N} \rangle$ is defined.
- Player I wins this run of the game if
  \[ \sum_{i=1}^{\infty} \epsilon_i / 2^i \in A \]
  otherwise Player II wins.

**Definition**

The set $A$ is **determined** if there is a winning strategy for one of the players in the game associated to $A$. 
The Axiom of Determinacy (AD)

**Definition (Mycielski-Steinhaus)**

**Axiom of Determinacy (AD):** Every set $A \subseteq \mathbb{R}$ is determined.

**Lemma (Axiom of Choice)**

*There is a set $A \subset \mathbb{R}$ such that $A$ is not determined.*

**Corollary**

*AD is false.*
### Projective Determinacy (PD)

**Definition**

**Projective Determinacy (PD):** Every projective set $A \subseteq \mathbb{R}$ is determined.

**Theorem**

Assume every projective set is determined.

1. (Mycielski-Steinhaus) *Every projective set has the property of Baire.*
2. (Mycielski-Swierczkowski) *Every projective set is Lebesgue measurable.*
3. (Moschovakis) *Every projective set $A \subseteq \mathbb{R} \times \mathbb{R}$ can be uniformized by a projective function.*

**Key questions**

Is PD even consistent and if consistent, is PD true?
Elementary embeddings

Definition

Suppose \( X \) and \( Y \) are transitive sets. A function \( j : X \rightarrow Y \) is an **elementary embedding** if for all logical formulas \( \varphi[x_0, \ldots, x_n] \) and all \( a_0, \ldots, a_n \in X \),

\[
(X, \in) \models \varphi[a_0, \ldots, a_n] \text{ if and only if } (Y, \in) \models \varphi[j(a_0), \ldots, j(a_n)]
\]

- Isomorphisms are elementary embeddings but the only isomorphisms of \( (X, \in) \) and \( (Y, \in) \) are trivial.

Lemma

*Suppose that\( j : X \rightarrow Y \) is an elementary embedding and that \( X \models \text{ZFC} \). Then the following are equivalent.*

1. \( j \) is not the identity.
2. There is an ordinal \( \beta \in X \) such that \( j(\beta) \neq \beta \).
Strong axioms of infinity: large cardinal axioms

**Basic template for large cardinal axioms**

A **cardinal** \( \kappa \) is a **large cardinal** if there exists an elementary embedding,

\[
j : V \rightarrow M
\]

such that \( M \) is a transitive class and \( \kappa \) is the least ordinal such that \( j(\alpha) \neq \alpha \).

- Requiring \( M \) be close to \( V \) yields a hierarchy of large cardinal axioms:
  - simplest case is where \( \kappa \) is a **measurable cardinal**.
  - \( M = V \) contradicts the Axiom of Choice.

The hierarchy of large cardinal axioms has emerged as the fundamental core of Set Theory.

- It is (empirically) a wellordered hierarchy and provides a calibration of the unsolvability of problems in Set Theory.
The validation of Projective Determinacy

Theorem (Martin-Steel)

Assume there are infinitely many Woodin cardinals. Then every projective set is determined.

Theorem

The following are equivalent.

1. Every projective set is determined.
2. For each $n < \omega$ there is a countable (iterable) model $M$ such that $M \models \text{ZFC} + \text{"There exist } n \text{ Woodin cardinals"}$.

PD is the missing (and true) axiom for $\langle \mathcal{P}(\mathbb{N}), \mathbb{N}, \cdot, +, \in \rangle$

- Is there such an axiom for $\mathcal{V}$ itself?