Algorithmic Fractal Dimensions

Jack H. Lutz
Iowa State University

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Lectures

1. Information and Dimensions, Classical and Algorithmic
2. Algorithmic Dimensions in Fractal Geometry
3. Mutual Dimensions and Finite-State Dimensions

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Lecture 1. Information and Dimensions, Classical and Algorithmic

Today's topics

- Shannon information (entropy)
- Algorithmic information (Kolmogorov complexity)
- Classical fractal dimensions
- Algorithmic fractal dimensions
- Dimensions of finite strings
- Dimension characterizations of Kolmogorov complexity

Shannon Information

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1. The Shannon self-information of $x \in X$ is

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   the value of $H_0(X)$ "apparent to x."

2. The Shannon entropy of (X, p) is

$$H(X, p) = E_p \log \frac{1}{p(x)} = \sum_{x \in X} p(x) \log \frac{1}{p(x)} .$$

Algorithmic Information (Kolmogorov Complexity)

All Turing machines here are self-delimiting: In addition to standard work tapes, they have a special program tape with a program tape head that is read-only and cannot move left.

- At start of computation with a program $\pi \in \{0,1\}^*$ the program tape contains

  $$\sqcup\, \pi \sqcup \sqcup \sqcup \ldots$$

  ($\sqcup$ = "blank") with the program tape head on the leftmost $\sqcup$.
- A computation's output (on, say, the first worktape) is undefined unless it halts with the program tape head on the last bit of π.

Algorithmic Information (Kolmogorov Complexity)

The Kolmogorov complexity of a string $x \in \{0,1\}^*$ is

$$K(x) = \min \{ |\pi| \mid \pi \in \{0,1\}^* \text{ and } U(\pi) = x \},$$

where U is a universal Turing machine.

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- x is "random" if $K(x) \approx |x|$.
- Routine coding extends this to $K(x)$ for $x \in \mathbb{N}$, $x \in \mathbb{Q}$, $x \in \mathbb{Q}^n$, etc.

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$$\mathbf{m}(x) = \sum_{\substack{\pi \in \{0,1\}^* \\ U(\pi) = x}} 2^{-|\pi|} .$$

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↗ Shannon self-information, using $\mathbf{m}$

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- Open ball of radius r about $x \in \mathcal{X}$:

$$B^o(x,r) = \{y \in \mathcal{X} \mid \rho(x,y) < r\}.$$

Hausdorff Measures in Metric Spaces

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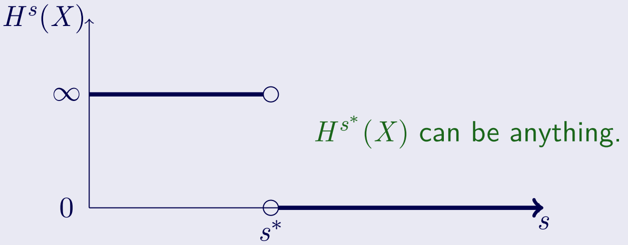
Definition (Hausdorff 1919)

Let ρ be a metric on $\mathcal{X}$, and let $X \subseteq \mathcal{X}$. The Hausdorff dimension of X with respect to ρ is

$$\dim^{(\rho)}(X) = \inf\{s \,|\, H^s(X) = 0\}\,.$$



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The s-dimensional packing (outer) ball measure of X is

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Definition (Tricot 1982, Sullivan 1984)

The packing dimension of X with respect to ρ is

$$\operatorname{Dim}^{(\rho)}(X) = \inf\{s \mid P^s(X) = 0\}.$$

Let Σ be an alphabet with $2 \leq |\Sigma| < \infty$.
A (Borel) probability measure on Σ^∞ is a function $\nu : \Sigma^* \to [0, 1]$ satisfying

$$\nu(\lambda) = 1, \quad \nu(w) = \sum_{a \in \Sigma} \nu(wa) \text{ for all } w \in \Sigma^* .$$

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Notation. μ is always the uniform probability measure on Σ^∞, i.e.,

$$\mu(w) = |\Sigma|^{-|w|} .$$

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We restrict attention to probability measures that are strongly positive, meaning that there exists $\delta > 0$ such that, for all $w \in \Sigma^*$ and $a \in \Sigma$, $\nu(wa) \geq \delta\nu(w)$.

Fractal Dimension in Sequence Spaces

The metric induced by a strongly positive probability measure ν on Σ^* is the function $\rho_\nu : \Sigma^\infty \times \Sigma^\infty \to [0, 1]$ given by

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$\dim^\nu(X)$ is also called the Billingsley dimension of X (Billingsley 1960).

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When $\nu = \mu$, we omit it from the terminology:

- The Hausdorff dimension of X is $\dim_H(X) = \dim^\mu(X)$.
- The packing dimension of X is $\dim_P(X) = \mathrm{Dim}^\mu(X)$.

Gale Characterizations

In a few minutes, we will define martingales, gales, and conditions for their success.

For the moment, martingales are strategies for betting on the successive symbols in a sequence $S \in \Sigma^\infty$, and one of these strategies succeeds on S if it makes an infinite amount of money betting on S.

Gales are generalized martingales that are no more powerful, but exhibit the martingales' success rates in a convenient form.

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J. Lutz and Mayordomo 2008:

Same for Billingsley dimensions.

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1. A ν-s-gale is a function $d : \Sigma^* \to [0, \infty)$ that satisfies

$$d(w)\nu(w)^s = \sum_{a \in \Sigma} d(wa)\nu(wa)^s$$

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4. A martingale is a 1-gale.

Gale Characterizations

Observation (J. Lutz 2000)

d is a ν-s-gale $\Leftrightarrow$ $d'(w) = \nu(w)^{s-1} d(w)$ is a ν-martingale.
$\therefore$ Gales are only a convenience.

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Definition

Let d be a ν-s-gale, and let $S \in \Sigma^\infty$.

1. d succeeds on S if $\limsup_{t\to\infty} d(S[0..t-1]) = \infty$.
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3. The success set of d is $S^\infty[d] = \{S \mid d \text{ succeeds on } S\}$.
4. The strong success set of d is $S^\infty_{\text{str}}[d] = \{S \mid d \text{ succeeds strongly on } S\}$.

Gale Characterizations

Theorem (J. Lutz and Mayordomo 2008)

Let ν be a strongly positive probability measure on Σ^∞, and let $X \subseteq \Sigma^\infty$.

1. *The Billingsley ν-dimension of X is*

$$\dim^\nu(X) = \inf \left\{ s \;\middle|\; \begin{array}{l} \textit{there is a } \nu\textit{-}s\textit{-gale } d \\ \textit{such that } X \subseteq S^\infty[d] \end{array} \right\}.$$

2. *The strong Billingsley ν-dimension of X is*

$$\mathrm{Dim}^\nu(X) = \inf \left\{ s \;\middle|\; \begin{array}{l} \textit{there is a } \nu\textit{-}s\textit{-gale } d \\ \textit{such that } X \subseteq S^\infty_{str}[d] \end{array} \right\}.$$

Gale Characterizations

Recall

1. The Hausdorff dimension of X is

$$\dim_H(X) = \dim^{\mu}(X)\,.$$

2. The packing dimension of X is

$$\dim_P(X) = \mathrm{Dim}^{\mu}(X)\,.$$

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We obtain Δ-algorithmic dimensions by requiring the gales in the gale characterizations to be Δ-computable.

Constructive Dimensions

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Definition

A ν-s-gale is constructive if it is lower semi-computable, i.e., if there is an exactly computable function $\hat{d} : \Sigma^* \times \mathbb{N} \to \mathbb{Q}$ with the following two properties.

- For all $w \in \Sigma^*$ and $t \in \mathbb{N}$, $\hat{d}(w, t) \leq \hat{d}(w, t + 1) < d(w)$.
- For all $w \in \Sigma^*$, $\lim_{t\to\infty} \hat{d}(w, t) = d(w)$.

Constructive Dimensions

Definition (J. Lutz and Mayordomo 2008, aided by a result of Fenner 2002)

Let ν be a strongly positive probability measure on Σ^∞, and let $X \subseteq \Sigma^\infty$.

1. The constructive ν-dimension of X is

$$\mathrm{cdim}^\nu(X) = \inf\left\{ s \;\middle|\; \begin{array}{l} \text{there is a constructive } \nu\text{-}s\text{-gale } d \\ \text{such that } X \subseteq S^\infty[d] \end{array} \right\}.$$

2. The constructive strong ν-dimension of X is

$$\mathrm{cDim}^\nu(X) = \inf\left\{ s \;\middle|\; \begin{array}{l} \text{there is a constructive } \nu\text{-}s\text{-gale } d \\ \text{such that } X \subseteq S^\infty_{\mathrm{str}}[d] \end{array} \right\}.$$

Constructive Dimensions

Definition (J. Lutz and Mayordomo 2008, aided by a result of Fenner 2002)

Let ν be a strongly positive probability measure on Σ^∞, and let $X \subseteq \Sigma^\infty$.

1. The constructive ν-dimension of X is

$$\mathrm{cdim}^\nu(X) = \inf\left\{ s \,\middle|\, \begin{array}{l} \text{there is a constructive } \nu\text{-}s\text{-gale } d \\ \text{such that } X \subseteq S^\infty[d] \end{array} \right\}.$$

2. The constructive strong ν-dimension of X is

$$\mathrm{cDim}^\nu(X) = \inf\left\{ s \,\middle|\, \begin{array}{l} \text{there is a constructive } \nu\text{-}s\text{-gale } d \\ \text{such that } X \subseteq S^\infty_{\mathrm{str}}[d] \end{array} \right\}.$$

We write $\mathrm{cdim}(X) = \mathrm{cdim}^\mu(X)$ and $\mathrm{cDim}(X) = \mathrm{cDim}^\mu(X)$.

Constructive Dimensions

A correspondence principle for an effective dimension is a theorem stating that, on sufficiently simple sets, the effective dimension coincides with its classical counterpart. (Terminology stolen from N. Bohr by J. Lutz.)

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Correspondence Principle for Constructive Dimension:

Theorem (Hitchcock 2002)

If $X \subseteq \Sigma^\infty$ is any union (not necessarily effective) of computably closed (i.e., Π^0_1) sets, then $\mathrm{cdim}(X) = \dim_H(X)$.

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Correspondence Principle for Constructive Strong Dimension: is false! (Conidis 2009)

Individual Sequences

Definition

Let ν be a probability measure on Σ^∞, and let $S \in \Sigma^\infty$.

1. The ν-dimension of S is $\dim^\nu(S) = \mathrm{cdim}^\nu(\{S\})$.
2. The strong ν-dimension of S is $\mathrm{Dim}^\nu(S) = \mathrm{cDim}^\nu(\{S\})$.

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Absolute Stability of Constructive Dimensions:

Theorem (Lutz and Mayordomo 2008, extending J. Lutz 2000)

If ν is a strongly positive, computable probability measure on Σ^∞, then, for all $X \subseteq \Sigma^\infty$,

$$\mathrm{cdim}^\nu(X) = \sup_{S \in X} \dim^\nu(S) \text{ and}$$

$$\mathrm{cDim}^\nu(X) = \sup_{S \in X} \mathrm{Dim}^\nu(S)\,.$$

(Contrast with the countable stability of classical dimensions)

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(Contrast with the countable stability of classical dimensions)

$\therefore$ Constructive dimensions are investigated in terms of individual sequences.

Individual Sequences

In general,

$$
\begin{array}{ccccccc}
0 & \leq & \dim_H(X) & \leq & \dim_P(X) & & \\
 & & \text{|}\!\wedge & & \text{|}\!\wedge & & \\
 & & \mathrm{cdim}(X) & \leq & \mathrm{cDim}(X) & \leq & 1\,.
\end{array}
$$

Individual Sequences

Definition (Martin-Löf 1966, Schnorr 1970)

A sequence $R \in \mathbf{C}$ is random if no constructive martingale succeeds on R.

Individual Sequences

If R is random (with respect to the uniform probability measure on $\mathbf{C}$, then

$$\dim(R) = \mathrm{Dim}(R) = 1\,.$$

Individual Sequences

What if R is random with respect to some other probability measure on $\mathbf{C}$?

Individual Sequences

Fix $\delta > 0$ and a bias sequence $\vec{\beta} = (\beta_0, \beta_1, \beta_2, \ldots)$ with each $\beta_i \in [\delta, 1 - \delta]$.

Definition

$$\mathcal{H}(\beta) = \beta \log \frac{1}{\beta} + (1 - \beta) \log \frac{1}{1 - \beta} = \text{ Shannon entropy}$$

$$H_n(\vec{\beta}) = \frac{1}{n} \sum_{i=0}^{n-1} \mathcal{H}(\beta_i)$$

$$H^-(\vec{\beta}) = \liminf_{n \to \infty} H_n(\vec{\beta}) \quad \text{lower average entropy}$$

$$H^+(\vec{\beta}) = \limsup_{n \to \infty} H_n(\vec{\beta}) \quad \text{upper average entropy}$$

Individual Sequences

Theorem (Athreya, Hitchcock, J. Lutz, and Mayordomo 2007)

Let $0 < \delta < \frac{1}{2}$*, and let* $\vec{\beta} = (\beta_0, \beta_1, \ldots)$ *be a computable bias sequence with each* $\beta_i \in \left[\delta, \frac{1}{2}\right]$*. For every* $\vec{\beta}$*-random sequence* R *we have*

$$\dim(R) = H^-(\vec{\beta}), \quad \mathrm{Dim}(R) = H^+(\vec{\beta})\,.$$

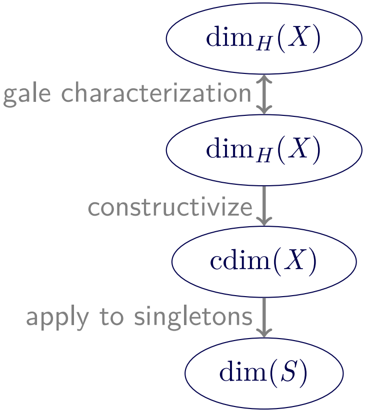
Dimensions of Finite Strings

Our next task: Extend Hausdorff dimension to define $\dim(x)$ for each $x \in \{0,1\}^*$.

Dimensions of Finite Strings

Our strategy:

For $X \subseteq \mathbf{C} = \{0,1\}^\infty$



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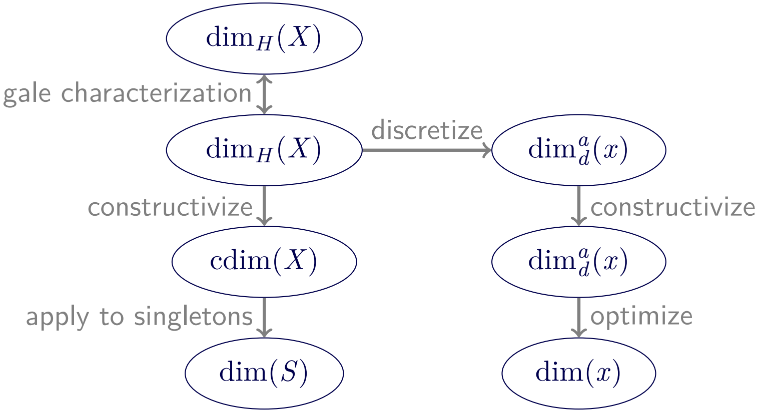

Dimensions of Finite Strings

Notation: $\mathcal{T} = \underbrace{\{0,1\}^*} \cup \underbrace{\{0,1\}^*\Box}$

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Dimensions of Finite Strings

Notation: $\mathcal{T} = \underbrace{\{0,1\}^*}_{\text{prefixes thereof}} \cup \underbrace{\{0,1\}^*\square}_{\text{terminated binary strings}}$

Definition

An s-termgale is a function $d : \mathcal{T} \to [0, \infty)$ satisfying

$$d(\lambda) \leq 1$$

and

$$d(w) \geq 2^{-s}(d(w0) + d(w1) + d(w\Box))$$

for all $w \in \{0,1\}^*$.

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Bets on the successive bits and termination of a finite string.

Dimensions of finite strings

Example

Define $d : \mathcal{T} \to [0, \infty)$ by

$$\begin{aligned} d(\lambda) &= 1 \\ d(w0) &= \frac{3}{2} d(w) \\ d(w1) &= d(w\square) = \frac{1}{4} d(w)\,. \end{aligned}$$

Dimensions of finite strings

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This is a 1-termgale. If $w \in \{0,1\}^n$ has n_0 0s and n_1 1s, then

$$
\begin{aligned}
d(w\Box) &= \left(\frac{3}{2}\right)^{n_0} \left(\frac{1}{4}\right)^{n_1+1} \\
&= 2^{n_0(1+\log 3)-2(n+1)} \,.
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$$d(w\square) = \left(\frac{3}{2}\right)^{n_0} \left(\frac{1}{4}\right)^{n_1+1}$$
$$= 2^{n_0(1+\log 3) - 2(n+1)} .$$

$\therefore$ If $n_0 \gg \frac{2}{1+\log 3}(n+1) \approx 0.77(n+1)$, then $d(w\square) \gg d(\lambda)$, even though d loses $\frac{3}{4}$ of its money when the $\square$ appears.

Dimensions of Finite Strings

Trivial observation: If

$$2^{-s|x|}d(x) = 2^{-s'|x|}d'(x) \qquad (*)$$

for all $x \in \mathcal{T}$, then d is an s-termgale $\Leftrightarrow$ d' is an s-termgale.

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A termgale is a family

$$d = \left\{ d^{(s)} \,\middle|\, s \in [0, \infty) \right\}$$

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d is completely determined by any one of its elements.

Dimensions of Finite Strings

Definition

Let d be a termgale, $a \in \mathbb{Z}^+$, and $w \in \{0,1\}^*$. The dimension of w relative to d at significance level a is

$$\dim_d^a(w) = \inf \left\{ s \,\middle|\, d^{(s)}(w\Box) > a \right\} .$$

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We have now discretized Hausdorff dimension.Constructivizing is easy:

Definition

A termgale d is constructive if $d^{(0)}$ is lower semicomputable.

Dimensions of Finite Strings

Now optimize

Definition

A constructive termgale $\tilde{d}$ is optimal if for every constructive termgale d there is a constant $\alpha > 0$ such that, for all $s \in [0, \infty)$ and $w \in \{0,1\}^*$,

$$\tilde{d}^{(s)}(w\square) \geq \alpha d^{(s)}(w\square).$$

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Theorem (J. Lutz 2003)

If $\tilde{d}$ is an optimal constructive termgale, then, for every constructive termgale d and every $a \in \mathbb{Z}^+$, there is a constant $\gamma \in [0, \infty)$ such that, for all $w \in \{0,1\}^*$,

$$\dim_{\tilde{d}}^{a}(w) \leq \dim_d(w) + \frac{\gamma}{1 + |w|} .$$

Dimensions of Finite Strings

Corollary

If d_1, d_2 are optimal constructive termgales and $a_1, a_2 \in \mathbb{Z}^+$, then there is a constant $\alpha \in [0, \infty)$ such that, for all $w \in \{0,1\}^*$,

$$\left|\dim_{d_1}^{a_1}(w) - \dim_{d_2}^{a_2}(w)\right| \leq \frac{\alpha}{1 + |w|}.$$

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Theorem (J. Lutz 2003)

There is an optimal constuctive termgale $\mathbf{d}_\square$. (Proof uses Levin's $\mathbf{m}$.)

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∴ It makes very little difference which optimal constructive termgale or which significance level we use.

Theorem (J. Lutz 2003)

There is an optimal constuctive termgale $\mathbf{d}_\square$. (Proof uses Levin's $\mathbf{m}$.)

Definition

The dimension of a string $w \in \{0,1\}^*$ is $\dim(w) = \dim_{\mathbf{d}_\square}^{1}(w)$.

Dimension and Kolmogorov Complexity

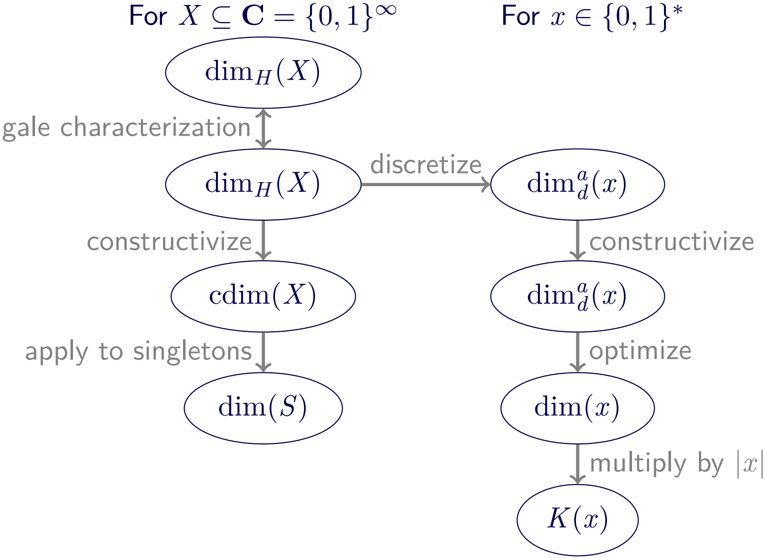
Theorem (J. Lutz 2003)

There is a constant $c \in \mathbb{N}$ *such that, for all* $x \in \{0,1\}^*$,

$$\big|K(x) - |x| \dim(x)\big| \leq c\,.$$

Dimension and Kolmogorov Complexity

Our strategy:



Dimension and Kolmogorov Complexity

$\therefore$ Up to constant additive terms,

$$K(x) = \log \frac{1}{\mathbf{m}(x)} = |x| \dim(x) \,.$$

The genius of Hausdorff, Shannon, and Kolmogorov: Their fundamentally different approaches to information, when constructivized and optimized (after discretizing $\dim_H$) lead to the same fundamental quantity, $K(x)$.

Thank you!