Algorithmic Fractal Dimensions

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Lectures

1. Information and Dimensions, Classical and Algorithmic
2. **Algorithmic Dimensions in Fractal Geometry**
3. Mutual Dimensions and Finite-State Dimensions

Lecture 2. Algorithmic Dimensions in Fractal Geometry

Today's topics

- Kolmogorov complexity characterizations of dimension
- Dimensions of points
- The Point-to-Set Principle
- Conditional Kolmogorov complexity in $\mathbb{R}^n$
- Kakeya sets in $\mathbb{R}^2$
- Dimensions of points on $y = mx + b$
- Generalized Furstenberg sets
- Intersections and products of fractals
- Pointwise dimensions

Kolmogorov Complexity Characterizations of Dimensions

Last time we saw that, up to additive constants,

$$K(x) = |x| \dim(x)$$

holds for all $x \in \{0,1\}^*$. Here is an infinitary version of this fact.

Theorem (J. Lutz and Mayordomo 2008)

If ν is a strongly positive, computable probability measure on Σ^∞, then, for all $S \in \Sigma^\infty$,

$$\dim^\nu(S) = \liminf_{m\to\infty} \frac{\mathrm{K}(S \upharpoonright m)}{\mathcal{I}_\nu(S \upharpoonright m)}$$

$$\mathrm{Dim}^\nu(S) = \limsup_{m\to\infty} \frac{\mathrm{K}(S \upharpoonright m)}{\mathcal{I}_\nu(S \upharpoonright m)},$$

where $\mathcal{I}_\nu(x) = \log \frac{1}{\nu(x)}$ is the Shannon ν-self-information of x.

Dimensions of Points

Work in Euclidean space $\mathbb{R}^n$.

The Kolmogorov complexity of a set $E \subseteq \mathbb{Q}^n$ is

$$K(E) = \min\{K(q) \mid q \in E\}\,.$$

(Shen and Vereschagin 2002)

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The Kolmogorov complexity of a set $E \subseteq \mathbb{R}^n$ is

$$K(E) = K(E \cap \mathbb{Q}^n).$$

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The Kolmogorov complexity of a set $E \subseteq \mathbb{R}^n$ is

$$K(E) = K(E \cap \mathbb{Q}^n)\,.$$

Note that

$$E \subseteq F \Rightarrow K(E) \geq K(F)\,.$$

Dimensions of Points

Let $x \in \mathbb{R}^n$ and $r \in \mathbb{N}$. The Kolmogorov complexity of x at precision r is

$$K_r(x) = K\left(B_{2^{-r}}(x)\right),$$

i.e., the number of bits required to specify some rational point $q \in \mathbb{Q}^n$ such that $|q - x| \leq 2^{-r}$.

Dimensions of Points

For $x \in \mathbb{R}^n$,

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Easy fact. $0 \leq \dim(x) \leq n$, and there are uncountably many points of each dimension in this interval.

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Old fact (J. Lutz '00 + Hitchcock '03). If $E \subseteq \mathbb{R}^n$ is a union of Π^0_1 sets, then

$$\dim_H(E) = \sup_{x \in E} \dim(x) .$$

classical Hausdorff (fractal) dimension

dimensions of individual points

$\therefore$ Dimensions of points are geometrically meaningful.

Point-to-Set Principle

Theorem (J. Lutz and N. Lutz, STACS '17)

For every $E \subseteq \mathbb{R}^n$,

$$\dim_H(E) = \min_{A \subseteq \mathbb{N}} \sup_{x \in E} \dim^A(x)\,.$$

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For every $E \subseteq \mathbb{R}^n$,

$$\dim_H(E) = \min_{A \subseteq \mathbb{N}} \sup_{x \in E} \dim^A(x) .$$

$\therefore$ In order to prove a lower bound

$$\dim_H(E) \geq \alpha ,$$

it suffices to show that

$$(\forall A \subseteq \mathbb{N})(\forall \varepsilon > 0)(\exists x \in E) \dim^A(x) \geq \alpha - \varepsilon$$

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or, if you're lucky, that

$$(\forall A \subseteq \mathbb{N})(\exists x \in E) \dim^A(x) \geq \alpha .$$

Point-to-Set Principle

Theorem (J. Lutz and N. Lutz, STACS '17)

For every $E \subseteq \mathbb{R}^n$,

$$\dim_P(E) = \min_{A \subseteq \mathbb{N}} \sup_{x \in E} \mathrm{Dim}^A(x)\,.$$

Conditional Kolmogorov Complexity in $\mathbb{R}^n$

Let $p \in \mathbb{Q}^m$ and $q \in \mathbb{Q}^n$. The conditional Kolomogorov complexity of p given q is

$$K(p|q) = \min\left\{|\pi| \,\middle|\, \pi \in \{0,1\}^* \text{ and } U(\pi, q) = p\right\}.$$

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Let $x \in \mathbb{R}^m$, $q \in \mathbb{Q}^n$, and $r \in \mathbb{N}$. The conditional Kolmogorov complexity of x given q at precision r is

$$\hat{K}_r(x|q) = \min \left\{ K(p|q) \,\middle|\, p \in \mathbb{Q}^m \cap B_{2^{-r}}(x) \right\}.$$

Conditional Kolmogorov Complexity in $\mathbb{R}^n$

Definition (J. Lutz and N. Lutz, STACS '17)

Let $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, and $r, s \in \mathbb{N}$. The conditional Kolmogorov complexity of x at precision r given y at precision s is

$$K_{r,s}(x|y) = \max\left\{\hat{K}_r(x|q) \,\middle|\, q \in \mathbb{Q}^n \cap B_{2^{-s}}(y)\right\}.$$

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For $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, and $r \in \mathbb{N}$,

$$K_r(x|y) = K_{r,r}(x|y).$$

Conditional Kolmogorov Complexity in $\mathbb{R}^n$

Chain rule for K_r:

$$K_r(x, y) = K_r(x|y) + K_r(y) + o(r) \,.$$

Easy fact. $K_r^y(x) \leq K_r(x|y) + o(r)$.

Kakeya Sets in $\mathbb{R}^2$

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Theorem (Davies 1971). Every Kakeya set in $\mathbb{R}^2$ has Hausdorff dimension 2.

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Kakeya Conjecture. Every Kakeya set in $\mathbb{R}^n$ has Hausdorff dimension n.

- An important open problem for $n \geq 3$.

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Davies's Theorem. Every Kakeya set in $\mathbb{R}^2$ has Hausdorff dimension 2.

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Davies's Theorem. Every Kakeya set in $\mathbb{R}^2$ has Hausdorff dimension 2.

Technical Lemma (J. Lutz and N. Lutz, STACS '17). Let $m \in [0,1]$ and $b \in \mathbb{R}$. For almost every $x \in [0,1]$,

$$\liminf_{r \to \infty} \frac{K_r(m,b,x) - K_r(b|m)}{r} \leq \dim(x, mx+b)\,.$$

Kakeya Sets in $\mathbb{R}^2$

Proof of Davies's Theorem (J. Lutz and N. Lutz, STACS '17).

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By the Point-to-Set Principle, fix $A \subseteq \mathbb{N}$ such that

$$\dim_H(K) = \sup_{z \in K} \dim^A(z)\,.$$

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Fix $m \in [0, 1]$ such that $\dim^A(m) = 1$.

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Fix a unit segment $L \subseteq K$ of slope m.

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Let (x_0, y_0) be the left endpoint of L.

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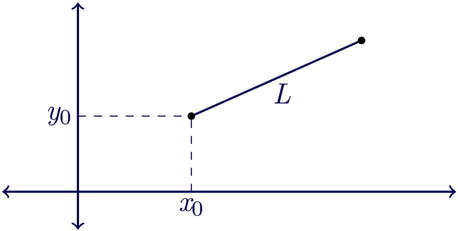

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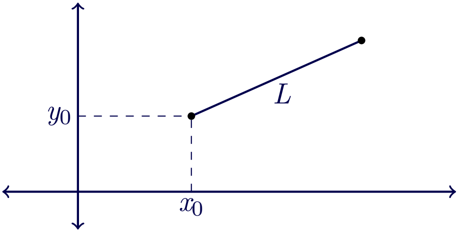
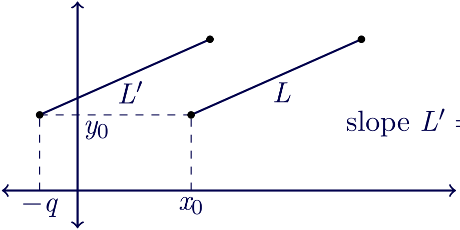


Let $q \in [x_0, x_0 + \frac{1}{2}]$.

Let L' be the unit segment of slope m whose left endpoint is $(x_0 - q, y_0)$.

Our picture is now



slope L' = slope $L = m$

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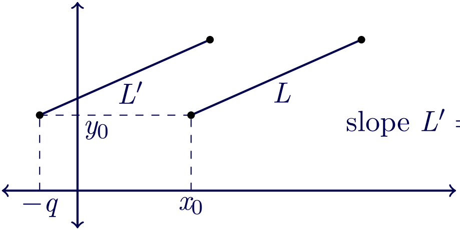

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Let $b = y_0 + qm$ be the y-intercept of L'.

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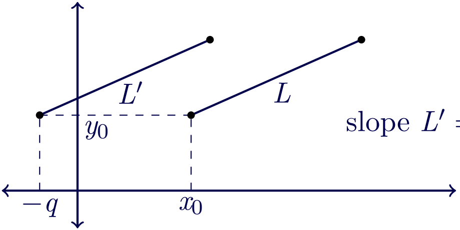

slope L' = slope $L = m$

Let $b = y_0 + qm$ be the y-intercept of L'.

By the Technical Lemma (relativized to A), fix $x \in [0, \frac{1}{2}]$ such that $\dim^{A,m,b}(x) = 1$ and

$$\liminf_{r\to\infty} \frac{K_r^A(m,b,x) - K_r^A(b|m)}{r} \leq \dim^A(x, mx+b) .$$

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By the Point-to-Set Principle it suffices to show that

$$\dim^A(x, mx + b) = 2\,.$$

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Why this suffices: Because

$$(x, mx + b) \in L'\,,$$

$$(x + q, mx + b) \in L \subseteq K\,,$$

and

$$\dim^A(x + q, mx + b) = \dim^A(x, mx + b)\,.$$

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Now $\dim^A(x, mx + b)$

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Tech Lemma $\quad \geq \liminf_{r\to\infty} \dfrac{K_r^A(m, b, x) - K_r^A(b|m)}{r}$

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Chain $= \liminf_{r\to\infty} \frac{K_r^A(m, b, x) - K_r^A(b, m) + K_r^A(m)}{r}$

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& \geq \liminf_{r\to\infty} \frac{K_r^{A,b,m}(x)}{r} + \liminf_{r\to\infty} \frac{K_r^A(m)}{r}
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& && = \dim^{A,b,m}(x) + \dim^A(m)
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	$= \dim^{A,b,m}(x) + \dim^A(m)$
	$= 2\,.$

□

Theorem (J. Lutz and Weihrauch 2008). Each of the sets

$$\mathsf{DIM}^{<1} = \{(x, y) \in \mathbb{R}^2 \mid \dim(x, y) < 1\},$$
$$\mathsf{DIM}^{>1} = \{(x, y) \in \mathbb{R}^2 \mid \dim(x, y) > 1\}$$

is totally disconnected.

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Theorem (J. Lutz and N. Lutz 2015). There is, in every direction in $\mathbb{R}^2$, a line missing every random point.

Theorem (J. Lutz and N. Lutz, STACS '17). Almost every point on every line $y = mx + b$ with random slope m has dimension 2.

Dimensions of Points on $y = mx + b$

Question (J. Lutz, early 2000s). Is there a line $y = mx + b$ on which every point has dimension 1?

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Theorem (N. Lutz and D. Stull, TAMC '17). For all $m, b, x \in \mathbb{R}$,

$$\dim(x, mx + b) \geq \dim^{m,b}(x) + \min\{\dim(m, b), \dim^{m,b}(x)\}\,.$$

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In particular, for almost every $x \in \mathbb{R}$,

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In particular, for almost every $x \in \mathbb{R}$,

$$\dim(x, mx + b) = 1 + \min\{\dim(m, b), 1\} .$$

Corollary. For every $m, b \in \mathbb{R}$ there exist $x_1, x_2 \in \mathbb{R}$ such that

$$\dim(x_1, mx_1 + b) - \dim(x_2, mx_2 + b) \geq 1 .$$

$\therefore$ The answer to the above question is "No!"

Generalized Furstenberg sets

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For $\alpha \in (0,1]$, a set $E \subseteq \mathbb{R}^2$ is α-Furstenberg if, for every $e \in S^1$ ($=$ the unit circle in $\mathbb{R}^2$), there is a line $\mathcal{L}_e$ in direction e such that $\dim_H(\mathcal{L}_e \cap E) \geq \alpha$.

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Definition (Molter and Rela 2012)

For $\alpha, \beta \in (0,1]$, a set $E \subseteq \mathbb{R}^2$ is (α, β)-generalized Furstenberg if there is a set $J \subseteq S^1$ such that $\dim_H(J) \geq \beta$ and, for every $e \in J$, there is a line $\mathcal{L}_e$ in direction e such that $\dim_H(\mathcal{L}_e \cap E) \geq \alpha$.

Generalized Furstenberg sets

Theorem (probably Furstenberg and Katznelson)

Fore $\alpha \in (0, 1]$, every α-Furstenberg set $E \subseteq \mathbb{R}^2$ satisfies

$$\dim_H(E) \geq \alpha + \max\left\{\frac{1}{2}, \alpha\right\} .$$

Note that Davies's theorem follows from the case $\alpha = 1$.

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Theorem (Molter and Rela 2012)

For $\alpha, \beta \in (0,1]$, every (α,β)-generalized Furstenberg set $E \subseteq \mathbb{R}^2$ satisfies

$$\dim_H(E) \geq \alpha + \max\left\{\frac{\beta}{2}, \alpha + \beta - 1\right\}.$$

Note that the previous theorem is the case $\beta = 1$.

Generalized Furstenberg sets

Theorem (N. Lutz and D. Stull, TAMC '17)

For $\alpha, \beta \in (0,1]$, every (α,β)-generalized Furstenberg set $E \subseteq \mathbb{R}^2$ satisfies

$$\dim_H(E) \geq \alpha + \min\{\beta, \alpha\}\,.$$

Note that this improves on the theorem of Molter and Rela exactly when $\alpha < 1$, $\beta < 1$, and $\beta < 2\alpha$. Hence it doesn't improve the bound on α-Furstenberg sets.

Generalized Furstenberg sets

Theorem (N. Lutz and D. Stull, TAMC '17)

For $\alpha, \beta \in (0,1]$, *every* (α, β)*-generalized Furstenberg set* $E \subseteq \mathbb{R}^2$ *satisfies*

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The proof is easy using the (nontrivial) $y = mx + b$ bound that we just saw and the Point-to-Set Principle.

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It is the first use of algorithmic fractal dimensions to prove a new theorem in classical fractal geometry!

Intersections and Products of Fractals

The following are fundamental, nontrivial, textbook theorems of fractal geometry.

Product Formula (Marstrand 1954). For all sets $E \subseteq \mathbb{R}^m$ and $F \subseteq \mathbb{R}^n$,

$$\dim_H(E \times F) \geq \dim_H(E) + \dim_H(F) \,.$$

Intersection Formula (Kahane 1986; Mattila 1984, 1985). For all Borel sets $E, F \subseteq \mathbb{R}^n$ and almost every $z \in \mathbb{R}^n$,

$$\dim_H(E \cap (F + z)) \leq \max\{0, \dim_H(E \times F) - n\} \,.$$

Note: The product formula was known earlier with extra assumptions on E and F. Marstrand deployed nontrivial machinery to prove it for arbitrary sets. Textbooks usually just prove it for Borel sets.

Intersections and Products of Fractals

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*The Intersection Formula holds for **all** sets $E, F \subseteq \mathbb{R}^n$.*

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This paper also uses a similar method to give a much simpler proof of the general Product Formula, along with analogous results for packing dimension.

Pointwise Dimensions

Classical fractal geometry has a pointwise notion of dimension.

An outer measure on $\mathbb{R}^n$ is a function $\nu : \mathcal{P}(\mathbb{R}^n) \to [0, \infty]$ satisfying

- $\nu(\emptyset) = 0$,
- $E \subseteq F \ \Rightarrow\ \nu(E) \leq \nu(F)$, and
- $E \subseteq \bigcup_{k=0}^{\infty} E_k \ \Rightarrow\ \nu(E) \leq \sum_{k=0}^{\infty} E_k$.

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An outer measure ν on $\mathbb{R}^n$ is

- finite if $\nu(\mathbb{R}^n) < \infty$, and
- locally finite if every $x \in \mathbb{R}^n$ has a neighborhood N with $\nu(N) < \infty$.

Pointwise Dimensions

Definition

Let ν be a locally finite outer measure on $\mathbb{R}^n$, and let $x \in \mathbb{R}^n$. The lower and upper pointwise dimensions of ν at x are

$$\dim_\nu(x) = \liminf_{r\to\infty} \frac{\log \frac{1}{\nu(B_{2^{-r}}(x))}}{r}$$

and

$$\mathrm{Dim}_\nu(x) = \limsup_{r\to\infty} \frac{\log \frac{1}{\nu(B_{2^{-r}}(x))}}{r},$$

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Are these in any way related to the algorithmic dimensions $\dim(x)$ and $\mathrm{Dim}(x)$?

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2. *For all $x \in \mathbb{R}^n$, $\dim(x) = \dim_\kappa(x)$ and $\mathrm{Dim}(x) = \mathrm{Dim}_\kappa(x)$.*
3. *This relativizes and interacts informatively with the Point-to-Set Principle.*

Thank you!