Algorithmic Fractal Dimensions

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Lectures

1. Information and Dimensions, Classical and Algorithmic
2. Algorithmic Dimensions in Fractal Geometry
3. **Mutual Dimensions and Finite-State Dimensions**

Lecture 3. Mutual Dimensions and Finite-State Dimensions

Today's topics

- Mutual dimensions
- Data processing inequalities
- Borel Normality
- Finite-state dimension
- Zeta-dimension
- Copeland-Erdős sequences
- Preserving finite-state dimension

Mutual Dimensions
(Case and J. Lutz 2015)

In both Shannon and algorithmic information theories, applications to communications and computation typically involve the mutual (shared) information between data objects. The following definition, which goes back to Kolmogorov, is analogous to that of Shannon mutual information.

Mutual Dimensions

Definition

The mutual information between $p \in \mathbb{Q}^m$ and $q \in \mathbb{Q}^n$ is

$$I(p : q) = K(p) - K(p|q) \, .$$

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Symmetry of information:

$$\begin{aligned} I(p : q) &= K(p) + K(q) - K(p, q) + O(1) \\ &= I(q : p) + O(1) \,. \end{aligned}$$

Mutual Dimensions

Definition

The mutual information between sets $E \subseteq \mathbb{R}^m$ and $F \subseteq \mathbb{R}^n$ is

$$I(E : F) = \min\{I(p : q) \,|\, p \in \mathbb{Q}^m \cap E \text{ and } q \in \mathbb{Q}^n \cap F\}.$$

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Definition

The mutual information between $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$ at precision $r \in \mathbb{N}$ is

$$I_r(x : y) = I\left(B_{2^{-r}}(x) : B_{2^{-r}}(y)\right).$$

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Definition

The mutual dimension between $x \in \mathbb{R}^m$ and $y \in \mathbb{R}^n$ is

$$\mathrm{mdim}(x : y) = \liminf_{r \to \infty} \frac{I_r(x : y)}{r} .$$

Data Processing Inequalities

- In Shannon information theory: If X, Y, and Z are ensembles and $f : X \to Y$, then

$$I(f(X); Z) \leq I(X; Z) .$$

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- In algorithmic information theory: If $f : \{0,1\}^* \to \{0,1\}^*$ is computable, then there is a constant $c_f \in \mathbb{N}$ such that, for all $x, z \in \{0,1\}^*$,

$$I(f(x) : z) \leq I(x : z) + c_f .$$

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- Today: If $f : \mathbb{R}^m \to \mathbb{R}^n$ is computable and Lipschitz, then, for all $x \in \mathbb{R}^m$ and $z \in \mathbb{R}^t$,

$$\mathrm{mdim}(f(x) : z) \leq \mathrm{mdim}(x : z) \,.$$

Data Processing Inequalities

Why/what Lipschitz??

Data Processing Inequalities

Why/what Lipschitz??

Example

Let $f : \mathbb{R} \to \mathbb{R}^2$ be computable and space-filling, in the sense that $[0,1]^2 \subseteq \mathrm{range}(f)$. (Examples are well known.) Choose $x \in \mathbb{R}$ such that $\dim(f(x)) = 2$, and let $z = f(x)$. Then

$$\begin{aligned} \mathrm{mdim}(f(x) : z) &= \mathrm{mdim}(f(x) : f(x)) \\ &= \dim(f(x)) \\ &= 2 \\ &> \dim(x) \\ &\geq \mathrm{mdim}(x : z)\,. \end{aligned}$$

Data Processing Inequalities

Definition

$f : \mathbb{R}^m \to \mathbb{R}^n$ is Lipschitz if there is a real number $c > 0$ such that, for all $x, x' \in \mathbb{R}^m$,

$$|f(x) - f(x')| \leq c|x - x'|\,.$$

Intuition: f is not so sensitive to its input that it can compress a great deal of "sparse" high-precision information about its input into "dense" lower-precision information about its output $f(x)$.

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To repeat, the data-processing inequality

$$\mathrm{mdim}(f(x) : z) \leq \mathrm{mdim}(x : z)$$

holds for all computable, Lipschitz $f : \mathbb{R}^m \to \mathbb{R}^n$.

Finite-State Dimension

Dai, Lathrop, J. Lutz, Mayordomo 2004

- A finite-state version of classical Hausdorff (fractal) dimension.
- For a sequence $S \in \Sigma^\infty$ (where $\Sigma = \{0, 1, \ldots, k-1\}$), $\dim_{\mathrm{FS}}(S) =$ "asymptotic density of finite-state information in S" $\in [0, 1]$.
- First defined using finite-state gamblers.

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- Equivalent definitions:

  Information-lossless finite-state compressors (DLMM 2004)
  Block-entropy rates (Bourke, Hitchcock, Vinodchandran 2006)
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Robust!

Normal Sequences

$\Sigma = \{0, 1, \ldots, k-1\}$
For $S \in \Sigma^\infty$, $w \in \Sigma^+$, $n \in \mathbb{Z}^+$,

$$\begin{aligned}\operatorname{freq}_n(w, S) &= \frac{\big|\{i < n \mid S[i..i+|w|-1] = w\}\big|}{n} \\ &= n^{\text{th}} \text{ frequency of } w \text{ in } S.\end{aligned}$$

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Definition (Borel 1909)

A sequence $S \in \Sigma^\infty$ is normal if

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Fact (Schnorr, Stimm 1972; BHV 2006)

S is normal $\Leftrightarrow \dim_{\mathrm{FS}}(S) = 1$.

Copeland-Erdős Sequences

For $n \in \mathbb{Z}^+$, $\sigma_k(n) = k$-ary expansion of n.

Definition

The k-ary Copeland-Erdős sequence of an infinite set

$$A = \{a_1 < a_2 < \ldots\} \subseteq \mathbb{Z}^+$$

is the sequence $\mathrm{CE}_k(A) = \sigma_k(a_1)\sigma_k(a_2)\sigma_k(a_3)\ldots$.

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Theorem (Champernowne 1933)

The decimal Champernowne sequence

$$\mathrm{CE}_{10}(\mathbb{Z}^+) = 12345678910111213\dots$$

is normal.

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The sequcence $\mathrm{CE}_{10}(\mathrm{PRIMES}) = 23571113171923\dots$ *is also normal.*

Copeland-Erdős Sequences

Theorem (Copeland & Erdős 1946)

For all $k \geq 2$, $\mathrm{CE}_k(\mathrm{PRIMES})$ *is normal.*

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For all $k \geq 2$, $\mathrm{CE}_k(\mathrm{PRIMES})$ *is normal.*

Outline of proof

1. For all sufficiently dense $A \subseteq \mathbb{Z}^+$, $\mathrm{CE}_k(A)$ is normal.
2. PRIMES is sufficiently dense. $\square$

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OBJECTIVE

Extend this to a quantitative lower-bound criterion for the finite-state dimensions of Copeland-Erdős sequences.

The Four Dimensions

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2. Finite-state strong dimension

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The Four Dimensions

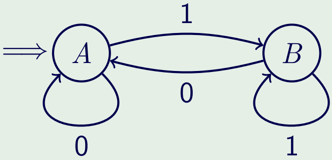
1. Finite-state dimension
2. Finite-state strong dimension
3. Zeta-dimension
4. Lower zeta-dimension

Finite-State Dimension

Dai, Lathrop, J. Lutz, Mayordomo 2004

Example

A 2-state gambler on the alphabet $\Sigma = \{0, 1\}$:



$$\beta(A)(0) = 0.3 \quad \beta(B)(0) = 0.7.$$

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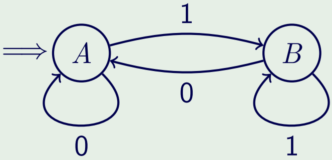

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$d_G(w) =$ capital G has after w if payoffs are fair.

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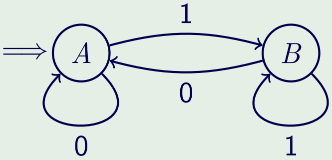

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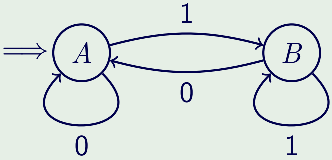

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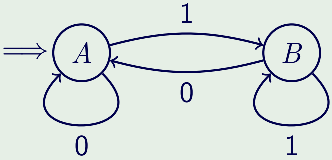

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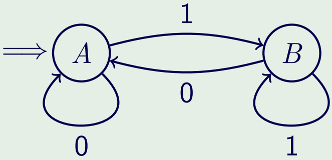

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$d_G(110) = 2(0.7)d_G(11) = 1.176$

$\vdots$

Finite-State Dimension

Definition

Let G be a finite-state gambler (FSG) over $\Sigma = \{0, 1, \ldots, k-1\}$, and let $s \in [0, \infty)$ be a "fairness parameter." The s-gale of G is the function $d_G^{(s)} : \Sigma^* \to [0, \infty)$ given by

$$d_G^{(s)}(\lambda) = 1$$

$$d_G^{(s)}(wa) = k^s d_G^{(s)}(w)\beta(\delta(w))(a)$$

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Definition

1. G s-succeeds on $S \in \Sigma^\infty$ if $\limsup_{n\to\infty} d_G^{(s)}(S \upharpoonright n) = \infty$.
2. G strongly s-succeeds on $S \in \Sigma^\infty$ if $\liminf_{n\to\infty} d_G^{(s)}(S \upharpoonright n) = \infty$.

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Definition

The finite-state dimension of a sequence $S \in \Sigma^\infty$ is

$$\dim_{\mathrm{FS}}(S) = \inf\{s \mid \exists \text{ an FSG that } s\text{-succeeds on } S\}.$$

Finite-State Strong Dimension

Athreya, Hitchcock, J. Lutz, Mayordomo 2007

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In general, $0 \le \dim_{\mathrm{FS}}(S) \le \mathrm{Dim}_{\mathrm{FS}}(S) \le 1$.

Zeta-Dimension

Invented many times!   "Discrete fractal dimension"

Definition

Let $A \subseteq \mathbb{Z}^+$.

- The A-zeta function $\zeta_A : [0,\infty) \to [0,\infty]$ is defined by

$$\zeta_A(s) = \sum_{n \in A} n^{-s}.$$

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Entropy characterization (Cahen 1894)

$$\mathrm{Dim}_\zeta(A) = \limsup_{n \to \infty} \frac{\log |A \cap \{1, \ldots, n\}|}{\log n}.$$

Zeta-Dimension

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The lower zeta-dimension of $A \subseteq \mathbb{Z}^+$ is

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The lower zeta-dimension of $A \subseteq \mathbb{Z}^+$ is

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Clearly, $0 \leq \dim_\zeta(A) \leq \mathrm{Dim}_\zeta(A) \leq 1$.

Extending Copeland & Erdős

Theorem (Gu, J. Lutz, and Moser 2007)

- *For every infinite* $A \subseteq \mathbb{Z}^+$*,*
  $\dim_{\mathrm{FS}}(\mathrm{CE}_k(A)) \geq \dim_\zeta(A)$ *and*
  $\mathrm{Dim}_{\mathrm{FS}}(\mathrm{CE}_k(A)) \geq \mathrm{Dim}_\zeta(A)$*.*

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- *For any four real numbers*

$$\begin{array}{ccccccc} 0 & \leq & \alpha & \leq & \beta & & \\ & & \rotatebox{90}{$\geq$} & & \rotatebox{90}{$\geq$} & & \\ & & \gamma & \leq & \delta & \leq & 1\,, \end{array}$$

there exists an infinite $A \subseteq \mathbb{Z}^+$ *with*

$$\begin{array}{cc} \dim_\zeta(A) = \alpha & \mathrm{Dim}_\zeta(A) = \beta \\ \dim_{\mathrm{FS}}(\mathrm{CE}_k(A)) = \gamma & \mathrm{Dim}_{\mathrm{FS}}(\mathrm{CE}_k(A)) = \delta\,. \end{array}$$

Extending Copeland & Erdős

Theorem (Gu, J. Lutz, and Moser 2007)

- *For every infinite* $A \subseteq \mathbb{Z}^+$,
  $\dim_{\mathrm{FS}}(\mathrm{CE}_k(A)) \geq \dim_\zeta(A)$ *and* $(*)$
  $\mathrm{Dim}_{\mathrm{FS}}(\mathrm{CE}_k(A)) \geq \mathrm{Dim}_\zeta(A)$.
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there exists an infinite $A \subseteq \mathbb{Z}^+$ *with*

$$\begin{array}{ll} \dim_\zeta(A) = \alpha & \mathrm{Dim}_\zeta(A) = \beta \\ \dim_{\mathrm{FS}}(\mathrm{CE}_k(A)) = \gamma & \mathrm{Dim}_{\mathrm{FS}}(\mathrm{CE}_k(A)) = \delta\,. \end{array}$$

$(*)$ implies the Copeland-Erdős theorem.

QUESTION

Which operations on sequences preserve normality?

Preserving Normality I: Subsequence Selection

Theorem (Wall 1949)

If S is normal and S' is a subsequence chosen by taking symbols at positions in an arithmetic progression, then S' is normal.

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Theorem (Agafonov 1968)

If S is normal and S' is a subsequence chosen using a regular language, then S' is normal.

Theorem (Merkle, Reimann 2003)

Subsequence selection using a context-free language — even a one-counter language — does not preserve normality.

Preserving Normality II: Real Arithmetic

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A real number may be normal in one base, but not in another.

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Theorem (Wall 1949)

If q is a non-zero rational, then, for every real number α, α is normal in base $k \Rightarrow q + \alpha$ and $q\alpha$ are normal base k.

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QUESTION

Which operations on sequences preserve finite-state dimension?

Observation: Every operation that preserves finite-state dimension preserves normality.

Preserving FSD I: Subsequence Selection

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$$S = b_0 0 b_1 0 b_2 0 b_3 \ldots ,$$

where $b_0 b_1 b_2 b_3 \ldots \in \{0, 1\}^\infty$ is normal, and consider

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Finite state dimension is not preserved by even the simplest subsequence selections.

Preserving FSD II: Real Arithmetic

Theorem (Doty, Lutz, and Nandakumar 2007)

For every base $k \geq 2$, every nonzero rational q, and every real number α, the base-k expansions of α, $q + \alpha$, and $q\alpha$ all have the same finite dimension and the same finite-state strong dimension.

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Extends Wall's 1949 theorem.
Gives a new proof of Wall's 1949 theorem.

Ingredients of Proof: Block-Entropy Rates

Notation:

- For $w, x \in \Sigma^+$,

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- For $S \in \Sigma^\infty$, $n \in \mathbb{Z}^+$, $0 < \ell < n$,

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Ingredients of Proof: Block-Entropy Rates

- For $S \in \Sigma^\infty$ and $\ell \in \mathbb{Z}^+$, the ℓ^{th} normalized lower and upper block entropy rates of S are

$$H_\ell^-(S) = \frac{1}{\ell \log k} \liminf_{n \to \infty} H\left(\pi_{S,n}^{(\ell)}\right),$$

$$H_\ell^+(S) = \frac{1}{\ell \log k} \limsup_{n \to \infty} H\left(\pi_{S,n}^{(\ell)}\right),$$

where

$$\begin{aligned} H(\pi) &= \text{ Shannon entropy of } \pi \\ &= \sum_w \pi(w) \log \frac{1}{\pi(w)}. \end{aligned}$$

Ingredients of Proof: Block-Entropy Rates

Theorem (Bourke, Hitchcock, Vinodchandran 2005)

For all $S \in \Sigma^\infty$,

$$\dim_{\mathrm{FS}}(S) = \inf_{\ell \in \mathbb{Z}^+} H_\ell^-(S)\,,$$

$$\mathrm{Dim}_{\mathrm{FS}}(S) = \inf_{\ell \in \mathbb{Z}^+} H_\ell^+(S)\,,$$

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The log-dispersion between $\mu, \pi \in \Delta_n$ is

$$\delta(\pi, \mu) = \log m\,,$$

where m is the least positive integer for which there is an $n \times n$ non-negative real matrix $A = (a_{ij})$ satisfying:

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- $A\pi = \mu$, i.e., $\sum_{j=1}^{n} a_{ij}\pi(j) = \mu(i)$ for all $1 \le i \le n$.
- No row or column of A contains more than m nonn-zero entries. ("Dispersion is limited by m.")

Ingredients of Proof: Logarithmic Dispersion

Definition

The normalized upper log-dispersion between sequences $S, T \in \Sigma^\infty$ is

$$\delta^+(S, T) = \limsup_{\ell \to \infty} \frac{1}{\ell \log k} \limsup_{n \to \infty} \delta\left(\pi_{S,n}^{(\ell)}, \pi_{T,n}^{(\ell)}\right) .$$

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1. $\delta^+(S,T) \geq 0$, with equality **if** (not iff)) $S = T$.
2. $\delta^+(S,T) = \delta^+(T,S)$.
3. $\delta^+(S,U) \leq \delta^+(S,T) + \delta^+(T,U)$.

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Main Technical Theorem:

Theorem (Doty, Lutz, and Nandadumar 2007)

$\dim_{\mathrm{FS}}$ and $\mathrm{Dim}_{\mathrm{FS}}$ are δ^+-contractive:

$$|\dim_{\mathrm{FS}}(S) - \dim_{\mathrm{FS}}(T)| \leq \delta^+(S, T),$$

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- May be more generally useful.
- Proof uses Schur concavity of Shannon entropy.

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Conjecture 2

Many of these more general theorems will drive the development of useful new methods.

Thank you!