The HOD Dichotomy

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Weak extender models and universality

Definition

Suppose λ is an uncountable cardinal.

- λ is a **singular cardinal** if there exists a cofinal set $X \subset \lambda$ such that $|X| < \lambda$.
- λ is a **regular cardinal** if there does not exist a cofinal set $X \subset \lambda$ such that $|X| < \lambda$.

Lemma (Axiom of Choice)

Every (infinite) successor cardinal is a regular cardinal.

Definition

Suppose λ is an uncountable cardinal. Then $\mathrm{cof}(\lambda)$ is the minimum possible $|X|$ where $X \subset \lambda$ is cofinal in λ.

- $\mathrm{cof}(\lambda)$ is always a regular cardinal.
- If λ is regular then $\mathrm{cof}(\lambda) = \lambda$.
- If λ is singular then $\mathrm{cof}(\lambda) < \lambda$.

Supercompactness

Definition

Suppose that κ is an uncountable regular cardinal and that $\kappa < \lambda$.

1. $\mathcal{P}_\kappa(\lambda) = \{\sigma \subset \lambda \,\big|\, |\sigma| < \kappa\}$.
2. Suppose that $U \subseteq \mathcal{P}\left(\mathcal{P}_\kappa(\lambda)\right)$ is an ultrafilter.
   - U is **fine** if for each $\alpha < \lambda$,
     $$\{\sigma \in \mathcal{P}_\kappa(\lambda) \,\big|\, \alpha \in \sigma\} \in U.$$
   - U is **normal** if for each function
     $$f : \mathcal{P}_\kappa(\lambda) \to \lambda$$
     such that
     $$\{\sigma \in \mathcal{P}_\kappa(\lambda) \,\big|\, f(\sigma) \in \sigma\} \in U,$$
     there exists $\alpha < \lambda$ such that
     $$\{\sigma \in \mathcal{P}_\kappa(\lambda) \,\big|\, f(\sigma) = \alpha\} \in U.$$

Definition

Suppose that κ is an uncountable regular cardinal. Then κ is a **supercompact cardinal** if for each $\lambda > \kappa$ there exists an ultrafilter U on $\mathcal{P}_\kappa(\lambda)$ such that:

1. U is κ-complete,
2. U is a normal fine ultrafilter.

Theorem (Solovay)

Suppose $\kappa < \lambda$ are uncountable regular cardinals and that U is a κ-complete normal fine ultrafilter on $\mathcal{P}_\kappa(\lambda)$.

- *Then there exists $Z \in U$ such that the function*

$$f(\sigma) = \sup(\sigma)$$

  is 1-to-1 on Z.

- The set Z does not depend on U.

Weak Extender Models

Definition

Suppose N is a transitive class, N contains the ordinals, and that N is a model of ZFC.

- Then N is a **weak extender model for the supercompactness of** δ iff for every $\gamma > \delta$ there exists a δ-complete normal fine ultrafilter U on $\mathcal{P}_\delta(\gamma)$ such that
  - $N \cap \mathcal{P}_\delta(\gamma) \in U$,
  - $U \cap N \in N$.

The Basic Thesis

If there is a generalization of L at the level of a supercompact cardinal then it should exist in a version which is a weak extender model for the supercompactness of some δ.

Elementary embeddings and weak extender models

Theorem (Weak Universality Theorem)

Suppose that N is a weak extender model for the supercompactness of δ, $\alpha > \delta$ is an ordinal, and that

$$j : N \cap V_{\alpha+1} \to N \cap V_{j(\alpha)+1}$$

is an elementary embedding such that $\delta \leq \mathrm{CRT}(j)$.

- *Then $j \in N$.*

- A much stronger version holds, which is the Universality Theorem.

Kunen's Theorem

Theorem (Kunen)

Suppose that λ is a cardinal.

- *Then there is no non-trivial elementary embedding*

$$j : V_{\lambda+2} \to V_{\lambda+2}.$$

Theorem

Let N be a weak extender model for the supercompactness of δ.

- *Then there is no nontrivial elementary embedding*

$$j : N \to N$$

  such that $\delta \leq \mathrm{CRT}(j)$.

- By the Weak Universality Theorem, for each cardinal λ, $j|(N \cap V_{\lambda+2}) \in N$.
- This implies there exists a cardinal $\lambda > \mathrm{CRT}(j)$ such that $j(\lambda) = \lambda$ and $j|(N \cap V_{\lambda+2}) \in N$.
- This contradicts Kunen's Theorem in N.

Theorem

Suppose that δ is a supercompact cardinal.

- *Then there is a weak extender model N for the supercompactness of δ, and a nontrivial elementary embedding $j : N \to N$.*

- By the Weak Universality Theorem, necessarily $\mathrm{CRT}(j) < \delta$.
- This shows that restriction on $\mathrm{CRT}(j)$ is necessary in the Weak Universality Theorem.

The Jensen Dichotomy

The Jensen Dichotomy Theorem

Theorem (Jensen)

Exactly one of the following holds.

(1) *For all singular cardinals γ, γ is a singular cardinal in L and*

$$\gamma^+ = (\gamma^+)^L.$$

- *L is* **close** *to V.*

(2) *Every uncountable cardinal is a regular limit cardinal in L.*

- *L is* **far** *from V.*

A strong version of Scott's Theorem:

Theorem (Silver)

Assume that there is a measurable cardinal.

- *Then L is far from V.*

Weak extender models for supercompactness

Theorem

Suppose that N is a weak extender model for supercompactness of δ, and that $\gamma > \delta$ is a singular cardinal. Then

- γ *is a singular cardinal in N,*
- $(\gamma^+)^N = \gamma^+$.

- There can be no (nontrivial) generalization of the Jensen Dichotomy Theorem to any weak extender model for supercompactness.
  - Weak inner models for supercompactness cannot be far from V

The HOD Dichotomy

Gödel's transitive class HOD

- Recall that a set M is transitive if every element of M is a subset of M.

Definition

HOD is the class of all sets X such that there exist $\alpha \in \mathrm{Ord}$ and $M \subset V_\alpha$ such that

1. $X \in M$ and M is transitive.
2. Every element of M is definable in V_α from ordinal parameters.

- (ZF) The Axiom of Choice holds in HOD.
- $L \subseteq \mathrm{HOD}$.

Lemma

Suppose $\alpha < \beta$ are ordinals and

$$X \subset \mathrm{HOD} \cap V_\alpha$$

is definable in V_β from ordinal parameters. Then $X \in \mathrm{HOD}$.

Stationary sets

Definition

Suppose λ is an uncountable regular cardinal.

1. A set $C \subset \lambda$ is **closed and unbounded** if C is cofinal in λ and contains all of its limit points below λ:
    - For all limit ordinals $\eta < \lambda$, if $C \cap \eta$ is cofinal in η then $\eta \in C$.
2. A set $S \subset \lambda$ is **stationary** if $S \cap C \neq \emptyset$ for all closed unbounded sets $C \subset \lambda$.

Example:

- Let $S \subset \omega_2$ be the set all ordinals α such that $\mathrm{cof}(\alpha) = \omega$.
    - S is a stationary subset of ω_2,
    - $\omega_2 \backslash S$ is a stationary subset of ω_2.

The Solovay Splitting Theorem

Theorem (Solovay)

Suppose that λ is an uncountable regular cardinal and that $S \subset \lambda$ is stationary.

- *Then there is a partition*

$$\langle S_\alpha : \alpha < \lambda \rangle$$

of S into λ-many pairwise disjoint stationary subsets of λ.

- But suppose $S \in \mathrm{HOD}$. Can one require

$$S_\alpha \in \mathrm{HOD}$$

for all $\alpha < \lambda$?

Definition

Let λ be an uncountable regular cardinal and let

$$S = \{\alpha < \lambda \,|\, \mathrm{cof}(\alpha) = \omega\}.$$

Then λ is ω-**strongly measurable in** HOD iff there exists $\kappa < \lambda$ such that:

1. $(2^\kappa)^{\mathrm{HOD}} < \lambda$,
2. there is no partition $\langle S_\alpha \mid \alpha < \kappa \rangle$ of S into stationary sets such that

   $$S_\alpha \in \mathrm{HOD}$$

   for all $\alpha < \lambda$.

A simple lemma

Suppose $\mathbb{B}$ is a complete Boolean algebra and γ is a cardinal.

- $\mathbb{B}$ is γ-cc if

  $$|\mathcal{A}| < \gamma$$

  for all $\mathcal{A} \subset \mathbb{B}$ such that $\mathcal{A}$ is an antichain:
  - $a \wedge b = 0$ for all $a, b \in \mathcal{A}$ such that $a \neq b$.

Lemma

Suppose that λ is an uncountable regular cardinal and that $\mathcal{F}$ is a λ-complete uniform filter on λ. Let

$$\mathbb{B} = \mathcal{P}(\lambda)/I$$

where I is the ideal dual to $\mathcal{F}$. Suppose that $\mathbb{B}$ is γ-cc for some γ such that $2^\gamma < \lambda$.

- *Then $|\mathbb{B}| \leq 2^\gamma$ and $\mathbb{B}$ is atomic.*

Lemma

Assume λ is ω-strongly measurable in HOD. Then

$$\mathrm{HOD} \models \lambda \textit{ is a measurable cardinal.}$$

Proof.

Let $S = \{\alpha < \lambda \mid (\mathrm{cof}(\alpha))^V = \omega\}$ and let

$\mathcal{F} = \{A \in \mathcal{P}(\kappa) \cap \mathrm{HOD} \mid S \backslash A$ is not a stationary subset of λ in $V\}$.

Thus $\mathcal{F} \in \mathrm{HOD}$ and in HOD, $\mathcal{F}$ is a λ-complete uniform filter on λ.

- Since λ is ω-strongly measurable in HOD, there exists $\gamma < \lambda$ such that in HOD:
  - $2^\gamma < \lambda$,
  - $\mathcal{P}(\lambda)/I$ is γ-cc where I is the ideal dual to $\mathcal{F}$.

Therefore by the simple lemma (applied within HOD), the Boolean algebra

$$(\mathcal{P}(\lambda) \cap \mathrm{HOD}) / I$$

is atomic. □

Extendible cardinals

Definition

Suppose that δ is a cardinal.

- Then δ is an **extendible cardinal** if for each $\lambda > \delta$ there exists an elementary embedding
$$\pi : V_{\lambda+1} \to V_{\pi(\lambda)+1}$$
such that $\mathrm{CRT}(\pi) = \delta$ and $\pi(\delta) > \lambda$.

Lemma

Suppose that δ is an extendible cardinal. Then

(1) *δ is a supercompact cardinal.*

(2) *δ is a limit of supercompact cardinals.*

Theorem

Suppose that δ is an extendible cardinal. Then the following are equivalent.

(1) HOD *is a weak extender model for the supercompactness of* δ.

(2) *There exists a regular cardinal* $\lambda > \delta$ *which is not* ω*-strongly measurable in* HOD.

Theorem (HOD Dichotomy Theorem)

Suppose that δ is an extendible cardinal. Then one of the following holds.

(1) *Every regular cardinal $\kappa \geq \delta$ is ω-strongly measurable in HOD. Further:*
   - HOD *is not a weak extender for the supercompactness of any λ.*
   - *There is no weak extender model N for the supercompactness of some λ such that $N \subseteq \mathrm{HOD}$.*

(2) *No regular cardinal $\kappa \geq \delta$ is ω-strongly measurable in HOD. Further:*
   - HOD *is a weak extender model for the supercompactness of δ.*
   - *Suppose γ is a singular cardinal and $\gamma > \delta$.*
     - *Then γ is singular cardinal in HOD and $\gamma^+ = (\gamma^+)^{\mathrm{HOD}}$.*

Theorem

Suppose that δ is an extendible cardinal.

- *Then δ is a measurable cardinal in HOD.*

Proof.

If δ is ω-strong measurable in HOD then δ is a measurable cardinal in HOD and so we can reduce to the case that

- δ is **not** ω-strongly measurable in HOD.

But then by HOD Dichotomy Theorem, HOD is a weak extender model for supercompactness of δ, and so δ is a supercompact cardinal in HOD.

The HOD Conjecture

Weak extender models for the measurability of δ

Definition

Suppose N is a transitive class, N contains the ordinals, and that N is a model of ZFC.

- Then N is a **weak extender model for the measurability of** δ iff there exists a δ-complete uniform ultrafilter U on δ such that

$$U \cap N \in N.$$

Theorem (Kunen 1971)

Suppose that δ is a measurable cardinal. Then there is a weak extender model for the measurability of δ such that

$$N \subset \mathrm{HOD}.$$

Speculation

Assume there is a supercompact cardinal (or more).

- *If there is an ultimate version of L and it can take the form of a weak extender model for δ is supercompact then the* HOD*-Dichotomy Theorem is* **not** *a genuine dichotomy theorem.*

- The generalization of Kunen's theorem to weak extender models for supercompactness would verify this.
- By the HOD Dichotomy Theorem this is actually an equivalence.

The HOD Hypothesis

Definition (The HOD Hypothesis)

There exists a proper class of regular cardinals λ which are not ω-strongly measurable in HOD.

1. *It is not known if there can exist 4 regular cardinals which are ω-strongly measurable in HOD.*
2. *Suppose γ is a singular cardinal of uncountable cofinality. It is not known if γ^+ can ever be ω-strongly measurable in HOD.*

Theorem (HOD Hypothesis)

Suppose that δ is an extendible cardinal.

- *Then HOD is a weak extender model for δ is supercompact.*

The HOD Conjecture

Definition (HOD Conjecture)

The theory

$$\text{ZFC} + \text{"There is a supercompact cardinal"}$$

proves the HOD Hypothesis.

- The HOD Conjecture is a number theoretic conjecture.
- If the HOD Conjecture holds then the theory

  $$\text{ZFC} + \text{"There is an extendible cardinal"}$$

  proves:
  - "HOD is a weak extender model for the supercompactness of some δ"

Applications of the HOD Conjecture in ZF

Theorem (ZF)

Assume the HOD Conjecture and that δ is an extendible cardinal.

- *Then for every cardinal $\lambda > \delta$, λ^+ is a regular cardinal.*

Theorem (ZF)

Assume the HOD Conjecture and that δ is an extendible cardinal.

- *Then for every regular cardinal $\lambda \geq \delta$, the Solovay Splitting Theorem holds at λ.*

Theorem (ZF)

Assume the HOD Conjecture and that δ is an extendible cardinal.

- *Then for every cardinal $\lambda > \delta$, there is no nontrivial elementary embedding $j : V_{\lambda+2} \to V_{\lambda+2}$.*

Assume the HOD Conjecture is true

Speculation on why the HOD Conjecture is true

If δ is an extendible cardinal then there must exist

$$N \subset V$$

such that:

- *N is a weak extender model for δ is supercompact.*
- *$N \subset \mathrm{HOD}$.*
- *N is a generalization of L.*

So:

- There should be an ultimate version of the axiom $V = L$.