Ultimate L

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The Clues

Clue 1

The First Clue

Assume there is a proper class of Woodin cardinals. Then there is an abstractly defined generalization of the projective sets:

- *These sets form a wellordered hierarchy under the rather fine notion of Borel complexity.*
- *The evidence suggests this hierarchy should somehow reflect the large cardinal hierarchy and the associated generalizations of L.*

Clue 2

The Second Clue: HOD Dichotomy Theorem

Assuming the existence of an extendible cardinal, HOD *is either very close to* V *or* HOD *is very far from* V*.*

- *The evidence suggests that the theorem is not a dichotomy theorem at all:*
  - HOD *should just be close to* V*.*

Reflection

A sentence φ is a Σ_2-sentence if it is of the form:

- *There exists an ordinal α such that $V_\alpha \models \psi$;*

for some sentence ψ.

In the context of ZFC:

- CH is expressible by a Σ_2-sentence.
- $(\neg\mathrm{CH})$ is expressible by a Σ_2-sentence.

Lemma

For each Σ_2-sentence φ, if $V \models \varphi$ then there exists a countable transitive set M such that

- *$M \models \mathrm{ZFC}\backslash\mathrm{Powerset}$,*
- *$M \models \varphi$.*

Defining the axiom $V = L$ without defining L

Suppose that M is a transitive set such that

- $M \models \mathrm{ZFC} \setminus \mathrm{Powerset}$.

Then

- $\mathrm{Ord}^M = M \cap \mathrm{Ord} = \sup\{a \in M \mid M \models \text{“}a \text{ is an ordinal”}\}$.

Lemma

The following are equivalent.

(1) $V = L$.

(2) *For each Σ_2-sentence φ, if $V \models \varphi$ then there exists a countable ordinal α such that $N \models \varphi$ where*
   - $N = \cap\{M \mid M \models \mathrm{ZFC} \setminus \mathrm{Powerset} \text{ and } \mathrm{Ord}^M = \alpha\}$.

- If one could find the correct **test models**:
  - This could be generalized to formulate the axiom $V = \mathrm{Ultimate}\text{-}L$
    - **without** having to refer to any construction of $\mathrm{Ultimate}\text{-}L$.

Recall: $L(A, \mathbb{R})$ where $A \subseteq \mathbb{R}$

Relativizing L to $A \subseteq \mathbb{R}$

Suppose $A \subseteq \mathbb{R}$. Define $L_\alpha(A, \mathbb{R})$ by induction on α by:

1. $L_0(A, \mathbb{R}) = \mathbb{R} \cup \{A\}$ (more precisely $L_0(A, \mathbb{R}) = V_{\omega+1} \cup \{A\}$),
2. (Successor case) $L_{\alpha+1}(A, \mathbb{R}) = \mathcal{P}_{\mathrm{Def}}(L_\alpha(A, \mathbb{R}))$,
3. (Limit case) $L_\alpha(A, \mathbb{R}) = \cup\{L_\beta(A, \mathbb{R}) \mid \beta < \alpha\}$.

- $L(A, \mathbb{R})$ is the class of all sets X such that $X \in L_\alpha(A, \mathbb{R})$ for some ordinal α.

- $\mathcal{P}(\mathbb{R}) \cap L_{\omega_1}(A, \mathbb{R})$ is the smallest σ-algebra containing A and closed under images by continuous functions $f : \mathbb{R} \to \mathbb{R}$.
- If $A \in L(\mathbb{R})$ then $L(A, \mathbb{R}) = L(\mathbb{R})$.

Some notation:

Definition

Θ is the supremum of the set of ordinals α for which there is a surjection

$$\rho : \mathbb{R} \to \alpha.$$

- Assuming the Axiom of Choice:
  - $\Theta = c^+$ where $c = 2^{\aleph_0} = |\mathbb{R}|$.

Suppose that $A \subset \mathbb{R}$. Then $\Theta^{L(A,\mathbb{R})}$ denotes Θ as computed in $L(A, \mathbb{R})$.

Theorem (Moschovakis)

Suppose that $A \subset \mathbb{R}$ and $L(A, \mathbb{R}) \models \mathrm{AD}$.

- *Then $\Theta^{L(A,\mathbb{R})}$ is a regular limit cardinal in $L(A, \mathbb{R})$.*

Ordinal games

Lemma (ZF)

There is a set $A \subset \omega_1^\omega$ which is not determined.

The generalization of AD to games on ordinals is inconsistent.

The correct generalization:

Suppose that $\lambda < \Theta$,

$$\pi : \lambda^\omega \to \omega^\omega$$

is continuous, and that $A \subset \omega^\omega$. Then $\pi^{-1}[A]$ is determined.

A refinement of the axiom AD: The axiom AD^+

Definition: AD^+ $(\mathrm{ZF} + \mathrm{DC})$

1. *Suppose $A \subset \mathbb{R}$. Then $A \in L(S, \mathbb{R})$ for some set $S \subset \mathrm{Ord}$.*
2. *Suppose that $\lambda < \Theta$,*
$$\pi : \lambda^\omega \to \omega^\omega$$
is continuous, and $A \subset \omega^\omega$. Then $\pi^{-1}[A]$ is determined.

Theorem

Suppose that $L(\mathbb{R}) \models \mathrm{AD}$. Then $L(\mathbb{R}) \models \mathrm{AD}^+$.

Theorem

Suppose that $A \subset \mathbb{R}$,
$$L(A, \mathbb{R}) \models \mathrm{AD}^+$$
and that $B \in \mathcal{P}(\mathbb{R}) \cap L(A, \mathbb{R})$.

- *Then $L(B, \mathbb{R}) \models \mathrm{AD}^+$.*

Martin-Steel Basis Theorem

A formula $\varphi(x)$ is a Σ_1-formula if it is of the form:

- *There exists a transitive set M such that $x \in M$ and $M \models \psi(x)$;*

for some formula $\psi(x)$.

Theorem (Martin-Steel)

Suppose that $L(\mathbb{R}) \models \mathrm{AD}$. Suppose that $\varphi(x, y)$ is a Σ_1-formula and

$$L(\mathbb{R}) \models \varphi[B, \mathbb{R}]$$

for some $B \in \mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})$.

- *Then there exists a set $B_0 \in \mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})$ such that*
  1. $L(\mathbb{R}) \models \varphi[B_0, \mathbb{R}]$.
  2. *Both B_0 and $\mathbb{R} \backslash B_0$ are Σ_1-definable in $L(\mathbb{R})$ from $\mathbb{R}$.*

The AD^+ Basis Theorem

Theorem

Suppose that $A \subset \mathbb{R}$ and $L(A, \mathbb{R}) \models \mathrm{AD}^+$. Suppose that $\varphi(x, y)$ is a Σ_1-formula and

$$L(A, \mathbb{R}) \models \varphi[B, \mathbb{R}]$$

for some $B \in \mathcal{P}(\mathbb{R}) \cap L(A, \mathbb{R})$.

- *Then there exists a set $B_0 \in \mathcal{P}(\mathbb{R}) \cap L(A, \mathbb{R})$ such that*
  (1) *$L(A, \mathbb{R}) \models \varphi[B_0, \mathbb{R}]$.*
  (2) *Both B_0 and $\mathbb{R} \backslash B_0$ are Σ_1-definable in $L(A, \mathbb{R})$ from $\mathbb{R}$.*

- If A is allowed as a parameter then the theorem is in general false.

Suslin sets and the uniformization problem

Definition

Suppose $A \subset \omega^\omega$. Then A is **Suslin** if there exists an ordinal λ, a continuous function

$$\pi : \lambda^\omega \to \omega^\omega,$$

and a closed set $C \subset \lambda^\omega$, such that $A = \pi[C]$.

- Define a set $A \subset \mathbb{R} \times \mathbb{R}$ to be Suslin if for some Borel bijection

$$\pi : \omega^\omega \to \mathbb{R} \times \mathbb{R},$$

the set $\pi^{-1}[A]$ is Suslin.

Lemma (ZF)

Suppose that

$$A \subset \mathbb{R} \times \mathbb{R}$$

and A is Suslin. Then A can be uniformized.

The Martin-Steel Suslin-Basis Theorem

Theorem (Martin-Steel)

Suppose that $L(\mathbb{R}) \models \mathrm{AD}$. *Suppose that* $\varphi(x)$ *is a* Σ_1*-formula and*

$$L(\mathbb{R}) \models \varphi[B, \mathbb{R}]$$

for some $B \in \mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})$.

- *Then there exists a set* $B_0 \in \mathcal{P}(\mathbb{R}) \cap L(\mathbb{R})$ *such that*
  1. $L(\mathbb{R}) \models \varphi[B_0, \mathbb{R}]$.
  2. *Both* B_0 *and* $\mathbb{R} \backslash B_0$ *are Suslin in* $L(\mathbb{R})$.

The AD^+ Suslin-Basis Theorem

The AD^+ Suslin-Basis Theorem

Suppose that $A \subset \mathbb{R}$ and $L(A, \mathbb{R}) \models \mathrm{AD}^+$. Suppose that $\varphi(x)$ is a Σ_1-formula and

$$L(A, \mathbb{R}) \models \varphi[B, \mathbb{R}]$$

for some $B \in \mathcal{P}(\mathbb{R}) \cap L(A, \mathbb{R})$.

- *Then there exists a set $B_0 \in \mathcal{P}(\mathbb{R}) \cap L(A, \mathbb{R})$ such that*
  (1) $L(A, \mathbb{R}) \models \varphi[B_0, \mathbb{R}]$.
  (2) *Both B_0 and $\mathbb{R} \backslash B_0$ are both Suslin in $L(A, \mathbb{R})$.*

Theorem

Suppose that $A \subset \mathbb{R}$ and $L(A, \mathbb{R}) \models \mathrm{AD}$.

- *Then the following are equivalent.*
  (1) $L(A, \mathbb{R}) \models \mathrm{AD}^+$.
  (2) *The AD^+ Suslin-Basis Theorem holds in $L(A, \mathbb{R})$.*

AD versus AD^+

Theorem ($\mathrm{ZF} + \mathrm{DC}$)

Suppose that AD *holds and that every set*

$$A \subset \mathbb{R} \times \mathbb{R}$$

can be uniformized. Then:

(1) *Every set is Suslin.*

(2) AD^+ *holds.*

Conjecture

Suppose $A \subset \mathbb{R}$. Then the following are equivalent.

1. $L(A, \mathbb{R}) \models \mathrm{AD}$.
2. $L(A, \mathbb{R}) \models \mathrm{AD}^+$.

Recall: the ultimate generalization of the projective sets

Definition (Feng-Magidor-Woodin)

A set $A \subseteq \mathbb{R}$ is **universally Baire** if for all topological spaces Ω and for all continuous functions $\pi : \Omega \to \mathbb{R}$, the preimage of A by π has the property of Baire in the space Ω.

Theorem

Suppose that there is a proper class of Woodin cardinals and that $A \subseteq \mathbb{R}$ is universally Baire. Then

(1) *Every set $B \in \mathcal{P}(\mathbb{R}) \cap L(A, \mathbb{R})$ is universally Baire.*

(2) $L(A, \mathbb{R}) \models \mathrm{AD}^+$.

$\mathrm{HOD}^{L(A,\mathbb{R})}$ and measurable cardinals

Theorem (Solovay)

Suppose that $A \subseteq \mathbb{R}$ and

$$L(A, \mathbb{R}) \models \mathrm{AD}.$$

Suppose $S \subset \omega_1$ and $S \in L(A, \mathbb{R})$.

- *Then there is a closed unbounded set $C \subset \omega_1$ such that:*
  - *Either $C \subset S$ or $C \cap S = \emptyset$.*
  - *$C \in L(A, \mathbb{R})$.*

Corollary

Suppose that there is a proper class of Woodin cardinals and that A is universally Baire.

- *Then ω_1 is a measurable cardinal in $\mathrm{HOD}^{L(A,\mathbb{R})}$.*

- $\mathrm{HOD}^{L(A,\mathbb{R})}$ denotes HOD as defined **within** $L(A, \mathbb{R})$.
  - $L(A, \mathbb{R}) \models \mathrm{ZF}$ but $L(A, \mathbb{R}) \not\models \mathrm{ZFC}$.
    - $\mathrm{HOD}^{L(A,\mathbb{R})} \models \mathrm{ZFC}$.

Recall:

Definition

Suppose that $A \subseteq \mathbb{R}$ is universally Baire.

Then $\Theta^{L(A,\mathbb{R})}$ is the supremum of the ordinals α such that there is a surjection, $\pi : \mathbb{R} \to \alpha$, such that $\pi \in L(A, \mathbb{R})$.

- $\Theta^{L(A,\mathbb{R})}$ is another measure of the complexity of A.

Lemma

Suppose there is a proper class of Woodin cardinals and that A, B are universally Baire. Then the following are equivalent:

(1) $A \in L(B, \mathbb{R})$.

(2) $\Theta^{L(A,\mathbb{R})} \leq \Theta^{L(B,\mathbb{R})}$.

$\mathrm{HOD}^{L(A,\mathbb{R})}$ and large cardinal axioms

Theorem

Suppose that there is a proper class of Woodin cardinals and that A is universally Baire.

- *Then $\Theta^{L(A,\mathbb{R})}$ is a Woodin cardinal in $\mathrm{HOD}^{L(A,\mathbb{R})}$.*

Here as before:

- $\mathrm{HOD}^{L(A,\mathbb{R})}$ denotes HOD as defined within $L(A,\mathbb{R})$.

$\mathrm{HOD}^{L(A,\mathbb{R})}$ and generalizations of L

Theorem (Steel)

Suppose that there is a proper class of Woodin cardinals and let $\delta = \Theta^{L(\mathbb{R})}$.

Then $\mathrm{HOD}^{L(\mathbb{R})} \cap V_\delta$ is a structural generalization of L.

- *which is constructed from a single predicate specifying a sequence partial extenders,*
  - *these are elementary embeddings $\pi : M \to N$ where M, N are transitive sets.*

Theorem

Suppose that there is a proper class of Woodin cardinals.

Then $\mathrm{HOD}^{L(\mathbb{R})}$ itself is a structural generalization of L.

- *But of a* **new and different** *type:*
  - *Constructed from* **two** *predicates.*

The axiom $V = \text{Ultimate-}L$

Assume there is a proper class of Woodin cardinals. Then for many universally Baire sets $A \subset \mathbb{R}$,

$$\mathrm{HOD}^{L(A,\mathbb{R})}$$

has been verified to be a structural generalization of L (of the new and different type).

- The natural conjecture is that must be true for **all** the universally Baire sets.

The axiom for $V = \text{Ultimate-}L$

- *There is a proper class of Woodin cardinals.*
- *For each Σ_2-sentence φ, if φ holds in V then there is a universally Baire set $A \subseteq \mathbb{R}$ such that*

$$\mathrm{HOD}^{L(A,\mathbb{R})} \models \varphi$$

- The restriction to Σ_2-sentences is necessary.

Consequences of $V = \mathrm{Ultimate}\text{-}L$

- One now can connect with AD^+-theory to obtain consequences of axiom $V = \mathrm{Ultimate}\text{-}L$.

Theorem ($V = \mathrm{Ultimate}\text{-}L$)

The Continuum Hypothesis holds.

- This follows from the AD^+ Suslin-Basis Theorem.

Theorem ($V = \mathrm{Ultimate}\text{-}L$)

*V is **not** a generic extension of any transitive class $N \subset V$.*

- Thus Cohen's method of forcing is completely useless in establishing independence in the context of the axiom $V = \mathrm{Ultimate}\text{-}L$.

Theorem ($V = \mathrm{Ultimate}\text{-}L$)

$V = \mathrm{HOD}$.

The Ultimate-L Conjecture

Question

Is there a generalization of Scott's Theorem to $V = \mathrm{Ultimate}\text{-}L$?

$\mathrm{Ultimate}\text{-}L$ Conjecture

(ZFC) *Suppose that δ is an extendible cardinal. Then there is a transitive class N such that:*

1. *N is a weak extender model for the supercompactness of δ.*
2. *$N \subseteq \mathrm{HOD}$.*
3. *$N \models$ "$V = \mathrm{Ultimate}\text{-}L$".*

- The conjecture implies there is no generalization of Scott's theorem to the case of $V = \mathrm{Ultimate}\text{-}L$.