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The accurate computation of the key properties of Markov chains and Markov Renewal Processes

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1. Introduction

Let $P = [p_{ij}]$ be the transition matrix of an irreducible, discrete time Markov chain (MC) $\{X_n\}$ $(n \geq 0)$ with finite state space $S = \{1, 2, \ldots, N\}$.

$$i.e.\ \ p_{ij} = P\{X_n = j \,|\, X_{n-1} = i\} \text{ for all } i, j \in S.$$

We are interested in developing accurate ways of finding two key properties of such chains:

(i) the stationary probabilites $\{\pi_j\}$, $(1 \leq j \leq N)$.

(ii) the mean first passage times $\{m_{ij}\}$,$(1 \leq i, j \leq N)$.

2. Stationary distributions

Let $\boldsymbol{\pi}^T = (\pi_1, \pi_2, \ldots, \pi_N)$ be the stationary prob. vector of the Markov chain with transition matrix $P = [p_{ij}]$.

We need to solve $\pi_j = \sum_{i=1}^{N} \pi_i p_{ij}$ with $\sum_{i=1}^{N} \pi_i = 1$,

i.e. $$\boldsymbol{\pi}^T (I - P) = \boldsymbol{0}^T \text{ with } \boldsymbol{\pi}^T \mathbf{e} = 1.$$

3. Mean first passage times

Let T_{ij} be the first passage time RV from state i to state j,

i.e. T_{ij} = min $\{n \geq 1$ such that $X_n = j$ given that $X_0 = i\}$.

T_{ii} is the first return to state i.

Let $m_{ij} = E[T_{ij} \mid X_0 = i]$, the mean first passage time from state i to state j.

The mean first passage times

Let $M = [m_{ij}]$ be the matrix of mean first passage times.

It is well known that

$$m_{ij} = 1 + \sum_{k \neq j} p_{ik} m_{kj}, \text{ with } m_{jj} = 1/\pi_j .$$

M satisfies the matrix equation

$$(I - P)M = E - PD,$$

where

$E = [1] = \mathbf{e}\mathbf{e}^T$, and

$D = M_d = [\delta_{ij} m_{ij}] = (\Pi_d)^{-1}$ (with $\Pi = \mathbf{e}\boldsymbol{\pi}^T$).

4. The GTH Algorithm

Let $P_N = \left[p_{ij} \right] = \left[p_{ij}^{(N)} \right]$ be the $N \times N$ transition matrix associated with the MC $\{X_k^{(N)}, k \geq 0\}$ with state space $S_N = \{1,...,N\}$ and transition probabilities

$p_{ij}^{(N)} = P\{X_{k+1}^{(N)} = j \mid X_k^{(N)} = i\}$.

The general approach is to start with the given MC $\{X_k, k \geq 0\}$, i.e. $X_k \equiv X_k^{(N)}$ on state space S_N, and reduce the state space by one state to S_{N-1} and repeat this succesively

$S_N = S_{N-1} \cup \{N\}$, $S_{N-1} = S_{N-2} \cup \{N-1\}, ..., S_2 = \{1,2\}$.

Once we get to two states we expand the state space one state at a time.

Assume that the initial MC with state space S_N is irreducible and that stationary probability vector is given by $\boldsymbol{\pi}^T = (\pi_1, \pi_2, ..., \pi_{N-1}, \pi_N)$

Let $\boldsymbol{\pi}^T = \boldsymbol{\pi}^{(N)T} = (\pi_1^{(N)}, \pi_2^{(N)}, ..., \pi_{N-1}^{(N)}, \pi_N^{(N)})$.

From the stationary equations $\boldsymbol{\pi}^{(N)T} = \boldsymbol{\pi}^{(N)T} P_N$ or in element form $\pi_j^{(N)} = \sum_{i=1}^{N} \pi_i^{(N)} p_{ij}^{(N)} \quad (j = 1, 2, ..., N)$

express $\pi_N^{(N)}$ in terms of $\pi_1^{(N)}, \ldots, \pi_{N-1}^{(N)}$:

$$\pi_N^{(N)} = \frac{\sum_{i=1}^{N-1} \pi_i^{(N)} p_{iN}^{(N)}}{\sum_{j=1}^{N-1} p_{Nj}^{(N)}}$$

and eliminate $\pi_N^{(N)}$ from the stationary equations.

Let $P_N = \begin{bmatrix} Q_{N-1}^{(N)} & \boldsymbol{p}_{N-1}^{(N)(c)} \\ \boldsymbol{p}_{N-1}^{(N)(r)T} & p_{NN}^{(N)} \end{bmatrix}$

Partition the stationary probability vector

$\boldsymbol{\pi}^{(N)T} = (\boldsymbol{\nu}^{(N-1)T},\ \pi_N^{(N)})$ where $\boldsymbol{\nu}^{(N-1)T} = (\pi_1^{(N)}, \pi_2^{(N)}, \ldots, \pi_{N-1}^{(N)})$

It is easily shown that

$\boldsymbol{\nu}^{(N-1)T}\left(I_{N-1} - P_{N-1}\right) = \boldsymbol{0}^T$, where $P_{N-1} = Q_{N-1}^{(N)} - \dfrac{\boldsymbol{p}_{N-1}^{(N)(c)}\boldsymbol{p}_{N-1}^{(N)(r)T}}{\boldsymbol{p}_{N-1}^{(N)(r)T}\boldsymbol{e}^{(N-1)}}$.

Let $P_{N-1} = \left[p_{ij}^{(N-1)}\right]$ then $p_{ij}^{(N-1)} = p_{ij}^{(N)} + \dfrac{p_{iN}^{(N)} p_{Nj}^{(n)}}{S(N)}$,

$$1 \le i \le N-1,\ 1 \le j \le N-1$$

Note that calculation of the $S(N)$ and the $p_{ij}^{(N-1)}$ do not involve subtractions.

P_{N-1} is a stochastic matrix with state space S_{N-1}

P_{N-1} is irreducible

$\boldsymbol{\nu}^{(N-1)T}$ is a scaled stationary prob vector of this $N-1$ state MC

$$\boldsymbol{\pi}^{(N-1)T} = (\pi_1^{(N-1)}, \pi_2^{(N-1)}, \ldots, \pi_{N-1}^{(N-1)}) \equiv \frac{1}{1-\pi_N^{(N)}} \boldsymbol{\nu}^{(N-1)T}$$

so that the first $N-1$ stationary probs of the N-state MC are scaled versions of the $N-1$ state MC.

Repeat this process reducing the state space from n to $n-1$, ($n = N, N-1, \ldots, 2$) with the resulting MC $\{X_k^{(n-1)}, k \geq 0\}$ having a stationary distribution that is a scaled version of the first $n-1$ components of the stationary distribution of the MC $\{X_k^{(n)}, k \geq 0\}$ with n states.

Thus if $P_n = \left[p_{ij}^{(n)} \right]$ and thus $P_{n-1} = \left[p_{ij}^{(n-1)} \right]$ then

$$p_{ij}^{(n-1)} = p_{ij}^{(n)} + \frac{p_{in}^{(n)} p_{nj}^{(n)}}{S(n)}, 1 \leq i \leq n-1, 1 \leq j \leq n-1;$$

where $S(n) = 1 - p_{nn}^{(n)} = \sum_{j=1}^{n-1} p_{nj}^{(n)}$.

The $p_{ij}^{(n-1)}$ can be interpreted as the transition prob from i to j of the MC *on* S_n *restricted to* S_{n-1}. ("*censored* MC")

The irreducibility of the initial MC $\{X_k^{(N)}, k \geq 0)\}$ leads to the irreducibility of the MC on S_n, $\{X_k^{(n)}, k \geq 0)\}$.

The process continues to $n = 2$, where we have the stochastic matrix $P_2 = \begin{bmatrix} p_{11}^{(2)} & p_{12}^{(2)} \\ p_{21}^{(2)} & p_{22}^{(2)} \end{bmatrix}$

The stationary distribution of this MC will be a scaled version of $\boldsymbol{\pi}^{(2)T} = (\pi_1^{(2)}, \pi_2^{(2)})$ or of (π_1, π_2).

The second stationary equation is $\pi_2 = \pi_1 p_{12}^{(2)} + \pi_2 p_{22}^{(2)}$

implying $\pi_2 = \pi_1 \dfrac{p_{12}^{(2)}}{S(2)}$.

$S(2) = 1 - p_{22}^{(2)} = \sum_{j=1}^{1} p_{2j}^{(2)} = p_{21}^{(2)} = \boldsymbol{p}_1^{(2)(r)T} e^{(1)}$.

We now proceed with increasing the state space.

In general, $\pi_n = \dfrac{\sum_{i=1}^{n-1} \pi_i p_{in}^{(n)}}{\sum_{i=1}^{n-1} p_{ni}^{(n)}} = \sum_{i=1}^{n-1} \pi_i \dfrac{p_{in}^{(n)}}{S(n)}$

GTH Algorithm

1. Start with a MC with N states and transition matrix $P_N = \left[p_{ij}^{(N)} \right]$.

2. Compute for $n = N, N–1, \ldots, ,3,$

$$p_{ij}^{(n-1)} = p_{ij}^{(n)} + \frac{p_{in}^{(n)} p_{nj}^{(n)}}{S(n)}, 1 \le i \le n-1, 1 \le j \le n-1; \text{ where}$$

$$S(n) = \sum_{j=1}^{n-1} p_{nj}^{(n)}.$$

3. Set $r_1 = 1$ and compute $r_n = \dfrac{\sum_{i=1}^{n-1} r_i p_{in}^{(n)}}{S(n)}$, for $n = 2,\ldots,N$.

4. Compute $\pi_i = \dfrac{r_i}{\sum_{j=1}^{N} r_j}$, $i = 1,2,\ldots, N$.

5. Mean First Passage Times via Extended GTH

We seek a computational procedure, utilising the GTH/State reduction procedure.

For a MC $\{X_n\}$ with N-states and transition matrix P, its mean first passage time matrix (MFPT) M satisfies

$$(I-P)M = E - PM_d$$

where $E = \left[1\right] = \boldsymbol{e}^{(N)}\boldsymbol{e}^{(N)T}$ and

$M_d = \left[\delta_{ij} m_{jj}\right] = diag(\pi_1, \pi_2, \ldots, \pi_N)$.

For a MRP $\{X_n, T_n\}$ the MFPT matrix satisfies

$$(I-P)M = \boldsymbol{\mu}^{(N)}\boldsymbol{e}^{(N)T} - P(M)_d.$$

M.R.P Primer

Let $\{(X_n, T_n)\}$, $(n \geq 0\}$, be a Markov renewal process (MRP) with state space S_N and semi-Markov kernel $Q(t) = \left[Q_{ij}(t) \right]$,

where $Q_{ij}(t) = P\{X_{n+1} = j,\ T_{n+1} - T_n \leq t \,|\, X_n = i\}$, $(i, j) \in S_N$.

X_n is the state at the n-th transition.

T_n is the time of the n-th transition.

Let $P = \left[p_{ij} \right]$ be the transition matrix of the embedded MC $\{X_n\}$, $(n \geq 0)$, $p_{ij} = Q_{ij}(+\infty)$ = $P\{X_{n+1} = j \,|\, X_n = i\}$.

$Q_{ij}(t) = p_{ij} F_{ij}(t)$ where $F_{ij}(t) = P\{T_{n+1} - T_n \leq t \,|\, X_n = i,\ X_{n+1} = j\}$.

$F_{ij}(t)$ is the distribution function of the "holding time" $T_{n+1} - T_n$ in state X_n until transition into state X_{n+1} given that the MRP makes a transition from X_n to X_{n+1}.

Let $\mu_{ij} = \int_0^\infty t\, dQ_{ij}(t)$ so that $\mu_{ij} = p_{ij} E[T_{n+1} - T_n \,|\, X_n = i,\, X_{n+1} = j]$.

Let $P^{(1)} = \left[\mu_{ij}\right]$ then $\quad (I - P)M = P^{(1)}E - PM_d$.

Let $\boldsymbol{\mu} = P^{(1)}\boldsymbol{e}$ then $\boldsymbol{\mu}^T = (\mu_1, \mu_2, \ldots, \mu_N)$ where $\mu_i = \sum_{j=1}^{N} \mu_{ij}$.

$\mu_i = E[T_{n+1} - T_n \,|\, X_n = i]$ is the "mean holding time in state i ".

Thus $P^{(1)}E = P^{(1)}\boldsymbol{e}\boldsymbol{e}^T = \boldsymbol{\mu}\boldsymbol{e}^T$

Let $\lambda_1 = \boldsymbol{\pi}^T \boldsymbol{\mu}$ "asymptotic mean increment" since for a M.R.P., $M_d = \lambda_1 (\Pi_d)^{-1}$ where $\Pi = \boldsymbol{e}\boldsymbol{\pi}^T$ implying $m_{jj} = \lambda_1 / \pi_j$.

Let us partition $M = M_N$ as $M_N = \begin{bmatrix} M_{N-1} & \boldsymbol{m}_{N-1}^{(N)(c)} \\ \boldsymbol{m}_{N-1}^{(N)(r)T} & m_{NN} \end{bmatrix}$

where

$M_{N-1} = \left[m_{ij} \right]$, $(1 \le i \le N-1,\ 1 \le j \le N-1)$,

$\boldsymbol{m}_{N-1}^{(N)(r)T} = (m_{N1}, m_{N2}, \ldots, m_{N,N-1})$ and

$\boldsymbol{m}_{N-1}^{(N)(c)T} = (m_{1N}, m_{2N}, \ldots, m_{N-1,N})$.

Let us also partition $\boldsymbol{\mu}^{(N)T} = (\mu_1^{(N)}, \ldots, \mu_{N-1}^{(N)}, \mu_N^{(N)}) = (\boldsymbol{\mu}_{N-1}^{(N)T}, \mu_N^{(N)})$

where $\boldsymbol{\mu}_{N-1}^{(N)T} = (\mu_1^{(N)}, \ldots, \mu_{N-1}^{(N)})$

If $\{(X_k^{(n)}, T_k^{(n)}),\ k \geq 0\}$ is a MRP with state space $S_n = \{1,\ldots,n\}$, transition matrix $P_n = \left[p_{ij}^{(n)}\right]$, MFPT matrix $M_n = \left[m_{ij}\right]$, $(1 \leq i \leq n, 1 \leq j \leq n)$, vector of mean holding times $\boldsymbol{\mu}^{(n)T} = (\mu_1^{(n)},\ldots,\mu_n^{(n)})$ then M_n satisfies

$$(I_n - P_n)M_n = \boldsymbol{\mu}^{(n)}\mathbf{e}^{(n)T} - P_n(M_n)_d.$$

Then $\{(X_k^{(n-1)}, T_k^{(n-1)}),\ k \geq 0\}$ is a MRP with state space $S_{n-1} = \{1,\ldots,n\}$, transition matrix $P_{n-1} = \left[p_{ij}^{(n-1)}\right]$, MFPT matrix $M_{n-1} = \left[m_{ij}\right]$, $(1 \leq i \leq n-1, 1 \leq j \leq n-1)$, vector of mean holding times $\boldsymbol{\mu}^{(n-1)T} = (\mu_1^{(n-1)},\ldots,\mu_n^{(n-1)})$ with M_{n-1} satisfying $(I_{n-1} - P_{n-1})M_n = \boldsymbol{\mu}^{(n-1)}\mathbf{e}^{(n-1)T} - P_{n-1}(M_{n-1})_d$,

where $p_{ij}^{(n-1)} = p_{ij}^{(n)} + \dfrac{p_{in}^{(n)} p_{nj}^{(n)}}{S(n)}$, $i, j = 1,..,n-1$,

and $\mu_i^{(n-1)} = \mu_i^{(n)} + \dfrac{p_{in}^{(n)} \mu_n^{(n)}}{S(n)}$, $i, j = 1,..,n-1$,

where $S(n) = 1 - p_{nn}^{(n)} = \sum_{j=1}^{n-1} p_{nj}^{(n)}$.

i.e. identical to the transition probabilities in the GTH algorithm.

Upon increasing the state space from S_{n-1} to S_n we wish to find the elements of M_n from M_{n-1}. Thus we need expressions for $\boldsymbol{m}_{n-1}^{(n)(c)}, \boldsymbol{m}_{n-1}^{(n)(r)}$, and m_{nn}.

(1) $m_{nn} = \dfrac{\lambda_1^{(N)}}{\pi_n^{(N)}}$ where $\lambda_1^{(N)} = \sum_{k=1}^{N} \pi_k^{(N)} \mu_k^{(N)}$,

(2) $m_{nn} = \dfrac{\lambda_1^{(n)}}{\pi_n^{(n)}}$ where $\lambda_1^{(n)} = \sum_{k=1}^{n} \pi_k^{(n)} \mu_k^{(n)}$,

(3) In the MC setting for $\{X_k^{(N)}, k \geq 0\}, m_{nn} = \dfrac{1}{\pi_n^{(N)}}$.

Further $m_{nn} = \mu_n^{(n)} + \sum_{k=1}^{n-1} p_{nk}^{(n)} m_{kn}, \quad n = 2,\ldots,N,$

where $m_{11} = \mu_1^{(1)}$,

For $\boldsymbol{m}_{n-1}^{(n)(r)}$

$$m_{nj} = \frac{\mu_n^{(n)} + \sum_{k=1, k\neq j}^{n-1} p_{nk}^{(n)} m_{kj}}{S(n)}, \; 1 \leq j \leq n-1, \; n \geq 2, \text{with } m_{21} = \frac{\mu_2^{(2)}}{S(n)}$$

It is more difficult to find $\boldsymbol{m}_{n-1}^{(n)(c)}$ i.e. m_{in} for $i = 1,..,n-1$.

In Hunter (2015) we show

$$m_{1n} = \frac{v_1^{(1,n)}}{R(1,n)}, \quad m_{in} = \frac{v_i^{(i,n)} + \sum_{k=1}^{i-1} q_{ik}^{(i,n)} m_{kn}}{R(i,n)}, i = 2,...,n-1,$$

where $q_{ik}^{(t-1,n)} = q_{ik}^{(t,n)} + \dfrac{q_{it}^{(t,n)} q_{tk}^{(t,n)}}{1 - q_{tt}^{(t,n)}}$, $i,k = 1,..,t-1$, $t = 2,...,n-1$,

with $q_{ik}^{(n-1,n)} = p_{ik}^{(n)}$, $i,k = 1,..,n-1$, $n = 2,...,N$,

and $v_i^{(t-1,n)} = v_i^{(t,n)} + \dfrac{q_{it}^{(t,n)} v_i^{(t,n)}}{1 - q_{tt}^{(t,n)}}$, $i.k = 1,..,t-1$, $t = 2,...,n-1$,

with $v_i^{(n-1,n)} = \mu_i^{(n)}$, $i = 1,..,n-1$, $n = 2,...,N$,

and $R(i,n) = 1 - q_{ii}^{(i,n)}$.

Have shown $R(i,n)$ can be expressed as a sum of terms for $i = n,\ n-1$. Conjectured true for all $i = 1,2,..,n$.

For $n = 2$: $$(I_2 - P_2)M_2 = \boldsymbol{\mu}^{(2)}\boldsymbol{e}^{(2)T} - P_2(M_2)_d$$

$$\begin{bmatrix} 1-p_{11}^{(2)} & -p_{12}^{(2)} \\ -p_{21}^{(2)} & 1-p_{22}^{(2)} \end{bmatrix} \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$$

$$= \begin{bmatrix} \mu_1^{(2)} & \mu_1^{(2)} \\ \mu_2^{(2)} & \mu_2^{(2)} \end{bmatrix} - \begin{bmatrix} p_{11}^{(2)} m_{11} & p_{12}^{(2)} m_{22} \\ p_{21}^{(2)} m_{11} & p_{22}^{(2)} m_{22} \end{bmatrix}$$

leading to

$$M_2 = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} = \begin{bmatrix} \dfrac{p_{21}^{(2)}\mu_1^{(2)} + p_{12}^{(2)}\mu_2^{(2)}}{p_{21}^{(2)}} & \dfrac{\mu_1^{(2)}}{p_{12}^{(2)}} \\ \dfrac{\mu_2^{(2)}}{p_{21}^{(2)}} & \dfrac{p_{21}^{(2)}\mu_1^{(2)} + p_{12}^{(2)}\mu_2^{(2)}}{p_{12}^{(2)}} \end{bmatrix}$$

General procedure for finding all the elements of *M*

Step 1: .Start with P_N and concentrate on finding only the expressions for m_{i1} for i = 1, 2, …, N.

i.e. if $P_N = \left[p_{ij}^{(N)} \right]$ carry out the GTH algorithm

For $n = N,\ N-1,\ \ldots, 3,$

let $p_{ij}^{(n-1)} = p_{ij}^{(n)} + \dfrac{p_{in}^{(n)} p_{nj}^{(n)}}{S(n)}, \quad 1 \le i \le n-1,\ 1 \le j \le n-1.$

Extend the algorithm (EGTH) to $\boldsymbol{\mu}^{(n)}$ to let

$$\mu_i^{(n-1)} = \mu_i^{(n)} + \frac{\mu_n^{(n)} p_{i,n}^{(n)}}{S(n)},\ (1 \le i \le n-1), \text{with } S(n) = \sum_{j=1}^{n-1} p_{nj}^{(n)}\ .$$

with $(\mu_1^{(N)}, \mu_2^{(N)} \ldots, \mu_N^{(N)}) = (1, 1, \ldots, 1).$

Let $m_{11} = \mu_1^{(1)} = \mu_1^{(2)} + \dfrac{p_{12}^{(2)}\mu_2^{(2)}}{p_{21}^{(2)}}$,

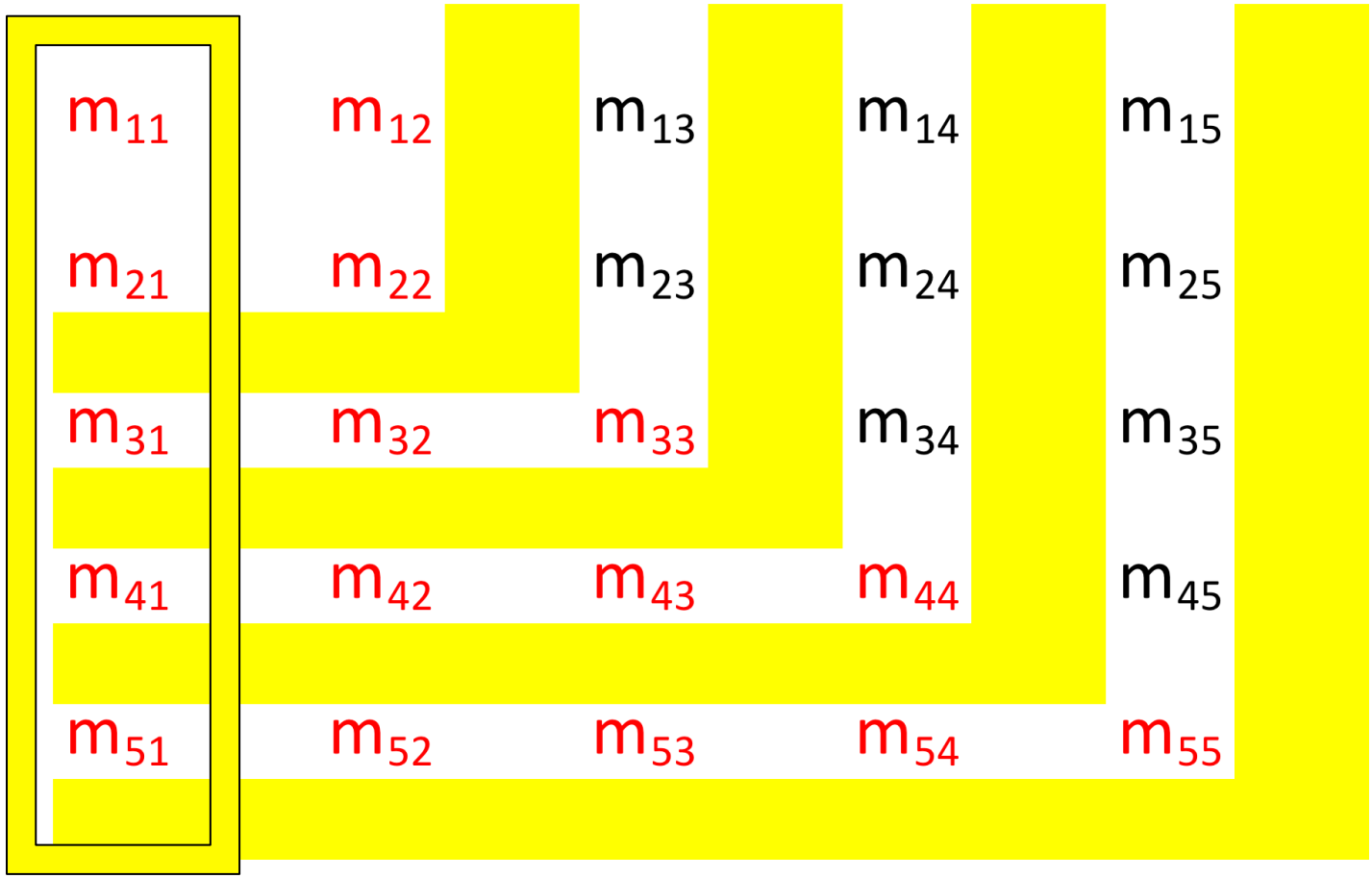
$$m_{21} = \frac{\mu_2^{(2)}}{S(2)},$$

$$m_{n1} = \frac{\mu_n^{(n)} + \sum_{k=2,}^{n-1} p_{nk}^{(n)} m_{k1}}{S(n)}, \; n = 3,\ldots,N.$$

This provides the entries of the first column of $M = \left[m_{ij} \right]$

i.e. $\boldsymbol{m}_N^{(1)(N)}$ since $M = \; (\boldsymbol{m}_N^{(1)(N)}, \boldsymbol{m}_N^{(2)(N)} \ldots, \boldsymbol{m}_N^{(N)(N)})$

where $\boldsymbol{m}_N^{(1)(N)T} = (m_{11}, m_{21}, \ldots, m_{N1})$.


m11 m12 m13 m14 m15
m21 m22 m23 m24 m25
m31 m32 m33 m34 m35
m41 m42 m43 m44 m45
m51 m52 m53 m54 m55

Step 2: Now reorder the rows of $P^{(N)}$ by moving the first column after the Nth column, followed by moving the first row to the last row.

$$P_N \equiv P_N^{(1)} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1,N-1} & p_{1,N} \\ p_{21} & p_{22} & & p_{2,N-1} & p_{2N} \\ & & & & \\ p_{N-1.1} & p_{N-1,2} & & p_{N-1,N-1} & p_{N-1,N} \\ p_{N1} & p_{N2} & & p_{N,N-1} & p_{NN} \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} p_{22} & p_{2,N-1} & p_{2N} & p_{21} \\ & & & \\ p_{N-1,2} & p_{N-1,N-1} & p_{N-1,N} & p_{N-1,1} \\ p_{N2} & p_{N,N-1} & p_{NN} & p_{N.1} \\ p_{12} & p_{1,N-1} & p_{1,N} & p_{11} \end{bmatrix} \equiv P_N^{(2)}$$

Step 2: Now reorder the rows of $P^{(N)}$ by moving the first column after the Nth column, followed by moving the first row to the last row.

$$P_N \equiv P_N^{(1)} = \begin{bmatrix} p_{11} & p_{12} & \cdots & p_{1,N-1} & p_{1,N} \\ p_{21} & p_{22} & & p_{2,N-1} & p_{2N} \\ & & & & \\ p_{N-1.1} & p_{N-1,2} & & p_{N-1,N-1} & p_{N-1,N} \\ p_{N1} & p_{N2} & & p_{N,N-1} & p_{NN} \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} p_{22} & p_{2,N-1} & p_{2N} & p_{21} \\ & & & \\ p_{N-1,2} & p_{N-1,N-1} & p_{N-1,N} & p_{N-1,1} \\ p_{N2} & p_{N,N-1} & p_{NN} & p_{N.1} \\ p_{12} & p_{1,N-1} & p_{1,N} & p_{11} \end{bmatrix} \equiv P_N^{(2)}$$

Step 3: Carry out the EGTH algorithm, as in Step 1, with $P_N = P_N^{(2)}$ to obtain the vector of MFPTs which we label as $\overline{\boldsymbol{m}}_N^{(2)(N)}$ where $\overline{\boldsymbol{m}}_N^{(2)(N)T} = (m_{22}, m_{32}, \ldots, m_{N2}, m_{12})$.

Step 4: Reorder $P_2^{(N)}$ as in step 2 to obtain $P_3^{(N)}$ and repeat Step 3 to obtain $\overline{\boldsymbol{m}}^{(3)(N)}$ where $\overline{\boldsymbol{m}}^{(3)(N)T} = (m_{33}, m_{43}, \ldots, m_{N3}, m_{13}, m_{23})$

Step k: Repeat as above with $P_k^{(N)}$ to obtain $\overline{\boldsymbol{m}}^{(k)(N)}$ where $\overline{\boldsymbol{m}}^{(k)(N)T} = (m_{kk}, m_{k+1,k}, \ldots, m_{N,k}, m_{1,k}, \ldots, m_{k-1,k})$ finishing with $P_N^{(N)}$ and $\overline{\boldsymbol{m}}^{(N)(N)}$ where $\overline{\boldsymbol{m}}^{(N)(N)T} = (m_{NN}, m_{1,N}, m_{2,N}, \ldots, m_{N-1,N})$

Step $N+1$: Let $\overline{M} = (\boldsymbol{m}_N^{(1)(N)}, \overline{\boldsymbol{m}}_N^{(2)(N)} \ldots, \overline{\boldsymbol{m}}_N^{(N)(N)})$

Finally reorder $\overline{M}$ to obtain $M = (\boldsymbol{m}_N^{(1)(N)}, \boldsymbol{m}_N^{(2)(N)} \ldots, \boldsymbol{m}_N^{(N)(N)})$

This can be carried out using the following MatLab procedure:

```
 end
    for col=1:m
        for row= 1:m
        P_new1(mod(row+m-2,m)+1,col)=P(row,col);
        end
    end
    for col=1:m
        for row= 1:m
       P_new2(row,mod(col+m-2,m)+1)=P_new1(row,col);
        end
    end
   P=P_new2;
    PP=P;
end
for col=1:m
    for row=1:m
        M1(mod(row+col-2,m)+1,col)=M(row,col);
    end
```

6. Test Problems

Introduced by Harrod & Plemmons (1984) and considered by others in different contexts.

TP1: The original transition matrix was not irreducible and was replaced (Heyman (1987), Heyman & Reeves (1989)) by

$$\begin{bmatrix} .1 & .6 & 0 & .3 & 0 & 0 \\ .5 & .5 & 0 & 0 & 0 & 0 \\ .5 & .2 & 0 & 0 & .3 & 0 \\ 0 & .7 & 0 & .2 & 0 & .1 \\ .1 & 0 & .8 & 0 & 0 & .1 \\ .4 & 0 & .4 & 0 & 0 & .2 \end{bmatrix}$$

TP2 (Also Benzi (2004))

$$\begin{bmatrix} .85 & 0 & .149 & .0009 & 0 & .00005 & 0 & .00005 \\ .1 & .65 & .249 & 0 & .00009 & .00005 & 0 & .00005 \\ .1 & .8 & .09996 & .0003 & 0 & 0 & .0001 & 0 \\ 0 & .0004 & 0 & .7 & .2995 & 0 & .0001 & 0 \\ .0005 & 0 & .0004 & .399 & .6 & .0001 & 0 & 0 \\ 0 & .00005 & 0 & 0 & .00005 & .6 & .2499 & .15 \\ .00003 & 0 & .00003 & .00004 & 0 & .1 & .8 & .0999 \\ 0 & .00005 & 0 & 0 & .00005 & .1999 & .25 & .55 \end{bmatrix}.$$

TP3

$$\begin{bmatrix} 0.999999 & 1.0\,E-07 & 2.0\,E-07 & 3.0\,E-07 & 4.0\,E-07 \\ 0.4 & 0.3 & 0 & 0 & 0.3 \\ 5.0\,E-07 & 0 & 0.999999 & 0 & 5.0\,E-07 \\ 5.0\,E-07 & 0 & 0 & 0.999999 & 5.0\,E-07 \\ 2.0\,E-07 & 3.0\,E-07 & 1.0\,E-07 & 4.0\,E-07 & 0.999999 \end{bmatrix}.$$

TP4 variants: **TP41** $\equiv \varepsilon$ = 1.0E-01, **TP42** $\equiv \varepsilon$ = 1.0E-03,

TP43 $\equiv \varepsilon$ = 1.0E-05, **TP44** $\equiv \varepsilon$ = 1.0E-07.

$$\begin{bmatrix} .1-\varepsilon & .3 & .1 & .2 & .3 & \varepsilon & 0 & 0 & 0 & 0 \\ .2 & .1 & .1 & .2 & .4 & 0 & 0 & 0 & 0 & 0 \\ .1 & .2 & .2 & .4 & .1 & 0 & 0 & 0 & 0 & 0 \\ .4 & .2 & .1 & .2 & .1 & 0 & 0 & 0 & 0 & 0 \\ .6 & .3 & 0 & 0 & .1 & 0 & 0 & 0 & 0 & 0 \\ \varepsilon & 0 & 0 & 0 & 0 & .1-\varepsilon & .2 & .2 & .4 & .1 \\ 0 & 0 & 0 & 0 & 0 & .2 & .2 & .1 & .3 & .2 \\ 0 & 0 & 0 & 0 & 0 & .1 & .5 & 0 & .2 & .2 \\ 0 & 0 & 0 & 0 & 0 & .5 & .2 & .1 & 0 & .2 \\ 0 & 0 & 0 & 0 & 0 & .1 & .2 & .2 & .3 & .2 \end{bmatrix}$$

9. Comparisons

We present comparisons for the test problems, the 4 algorithms (Standard, Simple, Perturbations and Extended GTH), under double precision, for the MFPT matrix M and compute the MAXIMUM ABSOLUTE ERRORS and the RESIDUAL ERRORS:

$$\text{MAX ABSOLUTE ERROR} = \max_{1 \le i \le m, 1 \le j \le m} \left| m_{ij} - \sum_{k \ne j} p_{ik} m_{kj} - 1 \right|$$

$$\text{RESIDUAL ERROR} = \sum_{i=1}^{m} \sum_{j=1}^{m} \left| m_{ij} - \sum_{k \ne j} p_{ik} m_{kj} - 1 \right|$$

7. Computation comparisons for MFPT

Using MatLab R2015b for a 64bit Mac under double precision we obtain the following for the Maximum Absolute errors for each algorithm and TP.

	M_Standard	M_Simple	M_Pert	M_EGTH
TP1	1.1369E-13	5.6843E-14	1.1369E-13	1.1369E-13
TP2	1.8190E-12	3.6380E-12	3.6380E-12	3.6380E-12
TP3	1.8626E-09	1.8626E-09	1.8626E-09	1.8626E-09
TP41	1.4211E-14	1.4211E-14	1.4211E-14	1.4211E-14
TP42	1.4552E-11	2.1828E-11	1.4552E-11	1.8190E-12
TP43	1.8626E-09	1.8626E-09	1.3970E-09	1.1642E-10
TP44	1.7881E-07	1.7881E-07	1.1921E-07	7.4506E-09

Using MatLab R2015b for a 64bit-Mac under double precision we obtain the following for the Residual errors for each algorithm and TP.

	M_Standard	M_Simple	M_Pert	M_EGTH
TP1	2.9465E-13	2.4603E-13	3.2123E-13	2.9177E-13
TP2	4.1670E-11	2.9660E-11	3.9564E-11	2.7776E-11
TP3	7.1555E-09	5.3028E-09	4.1790E-09	2.7940E-09
TP41	4.1922E-13	4.2477E-13	3.8503E-13	2.7337E-13
TP42	3.0482E-10	3.4037E-10	3.2139E-10	1.9142E-11
TP43	3.5818E-08	3.0308E-08	2.5198E-08	1.5717E-09
TP44	4.3242E-06	3.4264E-06	2.6729E-06	1.4156E-07