On Explicit Form of the Stationary Distributions for a Class of Bounded Markov Chains

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Model: an MC $\{X_n\}_{n\geq 0}$ on $[0,1]$ with the following dynamics.

Given: fixed d.f.'s F_L and F_R on $[0,1]$, a fixed measurable function $p : [0,1] \to [0,1]$.

Dynamics: given that $X_n = x \in [0,1]$, the next jump is to the left w.p. $p(x)$ or to the right w.p. $1 - p(x)$.

If to the left, the jump length is given by an independent random proportion $L_{n+1} \sim F_L$ of $[0,x]$. If to the right, the jump length is given by an ind't random proportion $R_{n+1} \sim F_R$ of interval $[x,1]$.

Motivation: some applications from Iacus & Negri (2003) and Ramli & Leng (2010).

Thus the evolution of the MC is given by:for $n = 0, 1, \ldots,$

$$X_{n+1} = X_n - X_n L_{n+1} I_{n+1} + (1 - X_n) R_{n+1} (1 - I_{n+1}),$$

where $X_0 = x_0 \in [0, 1]$, $I_{n+1} := \mathbf{1}_{\{U_{n+1} < p(X_n)\}}$, $\mathbf{1}_A$ being the indicator of the event A, and $\{L_n\}_{n \geq 1}$, $\{R_n\}_{n \geq 1}$ and $\{U_n\}_{n \geq 1}$ are independent sequences of i.i.d. random variables such that $L_n \sim F_L$, $R_n \sim F_R$, and $U_n \sim U[0, 1]$, the uniform law on $[0, 1]$.

Some history: Diaconis & Freedman (1999), case $p(x) \equiv p = \text{const}$, $F_L = F_R = U[0, 1]$.

Further publications: Stoyanov & Pirinsky (2000), Bialkowski & Wesolowski (2002), Stoyanov & Pacheco-Gonzalez (2008), Pacheco-Gonzalez (2009) ($F_L = F_R = \beta(1, z)$, $z > 0$, $p(x) \equiv x$; ergodicity + stationary law is $\beta(z, z)$), Ramli & Leng (2010) ($F_L = F_R = U[0, 1]$, $p(x)$ p/wise continuous)

What we do: in the case when

$$F_L = \beta(1, l), \quad F_R = \beta(1, r) \quad \text{for some } \; l, r > 0,$$

$p(x)$ p/w continuous satisfying a "natural condition", establish ergodicity and derive an *explicit formula for the stationary density.*

In particular, for $F_L = F_R = \beta(1, z)$, $z > 0$, and linear $p(x) = cx + (1 - b)(1 - x)$, $b, c \in (0, 1]$, the MC X is ergodic with stationary distribution $\beta(bz, cz)$.

Many of the existing results on the form of the stationary density for the above model are special cases of our theorem below.

The same approach: can compute (at least, numerically) the stationary pdf when $F_L = \beta(l_1, l_2)$, $F_R = \beta(r_1, r_2)$, $l_1, r_1 \in \mathbb{N}$, $l_2, r_2 > 0$, (or are finite mixtures of $\beta(1, z_i)$'s) by solving a two-point boundary value problem for a system of $l_1 + r_1$ ODEs.

Introduce conditions ensuring ergodicity.

[**E1**] For some $\delta \in (0, 1/2)$,

$$\sup_{x \in [0,\delta]} \max\{p(x), 1 - p(1-x)\} =: 1 - \varepsilon < 1.$$

Makes sense: when you are at the "left corner" $[0, \delta]$ *of your universe, there is a "positive chance" to go right:* p *is bounded away from 1. Similarly on the right.*

[**E2**] For some $\delta \in (0, 1/2)$,

$$\max\big\{F_L(1-\delta), F_R(1-\delta)\big\} =: 1 - \eta < 1.$$

This ensures the MC will eventually enter those corners.

[**E3**] There exist $\delta \in (0, 1/2)$ and $(s, t) \subset (\delta, 1-\delta)$, s.t. F_L and F_R have densities f_L and f_R on the intervals $(1-t-\delta, 1-s)$ and $(s-\delta, t)$, respectively. Moreover, for

$$g(z) := \min\left\{ \inf_{y\in[1-\delta,1]} f_L\left(\frac{y-z}{y}\right), \inf_{y\in[0,\delta]} f_R\left(\frac{z-y}{1-y}\right) \right\},$$

$z \in (s, t)$, one has

$$\gamma := \int_s^t g(z)\, dz > 0.$$

This is used for Harris condition (minorization of transition probabilities when starting from the "corners").

LEMMA 1 *If X satisfies conditions* [**E1**]–[**E3**] *with a common δ, then X is ergodic.*

STANDARD PROOF: Setting $\tau_B(x) := \inf\{k \geq 1 : X_k(x) \in B\}$, where $X_n(x) =$ value of the chain after n steps when $X_0 = x$, $B \in \mathcal{B}([0,1])$, it suffices[a] to show that there exists a $V \in \mathcal{B}([0,1])$, a probability measure φ on $([0,1], \mathcal{B}([0,1]))$, and a $q > 0$ such that

$$\text{(I}'\text{)} \quad \mathbb{P}(\tau_V(x) < \infty) = 1, \ \forall x \in [0,1],$$

$$\text{(I)} \quad \sup_{x \in V} \mathbb{E}\tau_V(x) < \infty,$$

$$\text{(II)} \quad \mathbb{P}(X_1(x) \in B) \geq q\varphi(B), \ \forall x \in V, \ \forall B \in \mathcal{B}([0,1])$$

(note that condition (II) implies aperiodicity of X).

[a] Theorems 1.3, 2.1 in A. Borovkov (1998).

Set $V := [0, \delta] \cup [1 - \delta, 1]$. By [**E2**], $\mathbb{P}(X_1(x) \in V) \geq \eta$, $x \in [0, 1]$, so the standard "worst scenario argument" yields that, for any $x_0 \in [0, 1]$ and $n \geq 1$,

$$\mathbb{P}(\tau_V(x_0) > n) = \mathbb{P}\left(\bigcap_{k=1}^{n} \{X_k(x_0) \notin V\} \right) \leq (1 - \eta)^n.$$

Hence (I′) and (I) hold true.

To verify (II), let $\varphi(B) := \dfrac{1}{\gamma}\displaystyle\int_{B\cap(s,t)} g(y)\,dy, \quad B \in \mathcal{B}([0,1])$. Using conditions [**E1**] and [**E3**], we have for $B \in \mathcal{B}([0,1])$ and $x \in [0,\delta]$,

$$\begin{aligned}
\mathbb{P}(X_1(x) \in B) &\geq \mathbb{P}(X_1(x) \in B \cap (s,t)) \\
&= (1-p(x)) \int_{B\cap(s,t)} \frac{1}{1-x} f_R\Big(\frac{y-x}{1-x}\Big) dy \\
&\geq \varepsilon \int_{B\cap(s,t)} g(y)\,dy \\
&= \varepsilon\gamma\varphi(B).
\end{aligned}$$

The same argument shows that the same bound holds when $x \in [1-\delta, 1]$. Therefore (II) is met with $q = \varepsilon\gamma$ and φ defined as above. The lemma is proved. $\square$

OK, now to explicit formulae for the stationary density $\pi(x)$ — which exists when F_L and F_R have densities and so the MC has transition density

$$f(x,y) = \begin{cases} \frac{1-p(x)}{1-x} f_R\left(\frac{y-x}{1-x}\right), & 0 \le x < y < 1, \\ \frac{p(x)}{x} f_L\left(\frac{x-y}{x}\right), & 0 < y < x \le 1, \end{cases}$$

and satisfies the usual integral equation

$$u(y) = \int_0^1 u(x) f(x,y)\, dx, \quad 0 < y < 1. \tag{1}$$

Consider *semidegenerate* $f(x,y)$: for some $N, M \geq 1$,

$$f(x,y) = \begin{cases} \sum_{i=1}^{N} a_i(y) b_i(x), & 0 \leq x < y < 1, \\ \sum_{j=1}^{M} c_j(y) d_j(x), & 0 < y \leq x \leq 1, \end{cases} \tag{2}$$

and the factors satisfy the following conditions:

[**C1**] all $a_i(y), c_j(y)$ are continuous on $(0,1)$;

[**C2**] all $b_i(x)$ are piecewise continuous on $[0,1)$;

[**C3**] all $d_j(x)$ are piecewise continuous on $(0,1]$.

Extending a result from Goldberg (1978)[a]: *If* $f(x,y)$ *is as above,* (1) *has an integrable on* $(0,1)$ *solution* u*, then:*

(i) $u(y)$ *is continuous on* $(0,1)$*;* **(ii)** *if*

$$\alpha_i(y) := \int_0^y b_i(x)u(x)\,dx, \quad i = 1,2,\ldots,N, \tag{3}$$

$$\beta_j(y) := \int_y^1 d_j(x)u(x)\,dx, \quad j = 1,2,\ldots,M, \tag{4}$$

then

$$u(y) = \sum_{i=1}^{N} a_i(y)\alpha_i(y) + \sum_{j=1}^{M} c_j(y)\beta_j(y), \quad 0 < y < 1. \tag{5}$$

[a] *Boundary and initial-value methods for solving Fredholm equations with semidegenerate kernels.* Assumed all the factors $a_i, \ldots, d_j$ are continuous on $[0,1]$.

Let S be the union of finite discontinuity sets for $b_i(x)$, $i = 1, \ldots, N$, and $d_j(x)$, $j = 1, \ldots, M$. Then

$$\alpha_i'(y) = b_i(y)u(y), \quad y \in (0,1)\backslash S, \tag{6}$$

$$-\beta_j'(y) = d_j(y)u(y), \quad y \in (0,1)\backslash S, \tag{7}$$

$$\alpha_i(0+) = 0, \quad i = 1, 2, \ldots, N, \quad \beta_j(1-) = 0, \quad j = 1, \ldots, M. \tag{8}$$

(iii) *Conversely, let $\alpha_i(y)$, $i = 1, 2, \ldots, N$, and $\beta_j(y)$, $j = 1, \ldots, M$, be continuous solutions of (5)–(8) such that $u(y)$ is integrable on $(0,1)$. Then $u(y)$ given by (5) is a solution of (1).*

Proof. From the assumption & formula for $f(x,y)$,

$$u(y) = \int_0^1 u(x) f(x,y)\, dx$$
$$= \sum_{i=1}^{N} a_i(y) \int_0^y b_i(x) u(x)\, dx + \sum_{j=1}^{M} c_j(y) \int_y^1 d_j(x) u(x)\, dx.$$

From here infer (i), (ii).

In (iii), assumed continuity of α_i and β_j solving (6), (7) means theses functions have the form (3), (4), resp. Substituting the integrals in (5) leads to the above equalities, q.e.d. $\square$

Now suppose $f_L(y) = l(1-y)^{l-1}$, $f_R(y) = r(1-y)^{r-1}$, $y \in (0,1)$. Then $f(x,y)$ is semidegenerate (2) with $N = M = 1$ and

$$a_1(y) = r(1-y)^{r-1}, b_1(x) = \frac{1-p(x)}{(1-x)^r}, c_1(y) = ly^{l-1}, d_1(x) = \frac{p(x)}{x^l}.$$

THEOREM 1 *Assume that $p(x)$ is piecewise continuous on $[0,1]$ and satisfies* [**E1**], $F_L = \beta(1,l)$, $F_R = \beta(1,r)$, $l, r > 0$. *Then X is ergodic with stationary density*

$$\pi(x) = Cx^l \left(\frac{r}{1-x} + \frac{l}{x} \right) \exp\left(-r \int_{1/2}^{x} \frac{p(t)}{1-t}\, dt - l \int_{1/2}^{x} \frac{p(t)}{t}\, dt \right),$$

where $C > 0$ is a constant such that $\int_0^1 \pi(x)\, dx = 1$.

Proof. Conditions [**E2**], [**E3**] satisfied $\Rightarrow$ ergodicity. Conditions [**C1**]–[**C3**] are also satisfied. All is good.

Clearly, S = set of discontinuities of $p(x)$ and $p(1-x)$.

Substituting the expressions for $\alpha \equiv \alpha_1$ etc into (6), (7) with u given by (5), we get, for $y \in (0,1)\backslash S$,

$$\alpha'(y)(1-y)^r = (1-p(y))\left(r(1-y)^{r-1}\alpha(y) + ly^{l-1}\beta(y)\right),$$
$$-\beta'(y)y^l = p(y)\left(r(1-y)^{r-1}\alpha(y) + ly^{l-1}\beta(y)\right).$$

Add the equations:

$$(\alpha(y)(1-y)^r)' = (\beta(y)y^l)', \quad y \in (0,1)\backslash S.$$

Integrate, assuming that α and β are continuous, and use the boundary condition: $\alpha(y)(1-y)^r = \beta(y)y^l$.

Substituting this identity into the ODE with β' leads to

$$\beta'(y) = -\beta(y)p(y)\left(\frac{r}{1-y} + \frac{l}{y}\right), \quad y \in (0,1)\backslash S,$$

with the general solution

$$\beta(y) = C_2 \exp\left(-r\int_{1/2}^{y} \frac{p(t)}{1-t}\,dt - l\int_{1/2}^{y} \frac{p(t)}{t}\,dt\right), \quad y \in (0,1).$$

Now, from the identity,

$$\alpha(y) = C_2 \frac{y^l}{(1-y)^r} \exp\left(-r\int_{1/2}^{y} \frac{p(t)}{1-t}\,dt - l\int_{1/2}^{y} \frac{p(t)}{t}\,dt\right).$$

Both α and β are continuous, so can use (iii), which proves the assertion of Theorem 1. $\square$

Remark 1 Can also to compute (at least, numerically) the stationary density of X when $p(x)$ satisfies [**E1**], while $F_L = \beta(l_1, l_2)$, $l_1 \in \mathbb{N}$, $l_2 > 0$, and $F_R = \beta(r_1, r_2)$, $r_1 \in \mathbb{N}$, $r_2 > 0$:

$$f_L\left(\frac{x-y}{x}\right) = \frac{1}{B(l_1, l_2)}\left(\frac{x-y}{x}\right)^{l_1-1}\left(\frac{y}{x}\right)^{l_2-1}, \quad 0 < y < x \leq 1,$$

$$f_R\left(\frac{y-x}{1-x}\right) = \frac{1}{B(r_1, r_2)}\left(\frac{y-x}{1-x}\right)^{r_1-1}\left(\frac{1-y}{1-x}\right)^{r_2-1}, \; 0 \leq x < y < 1.$$

So the transition probabilities are semidegenerate with $N = r_1$, $M = l_1$ (expand $(y-x)^s$). It remains to solve the two-point boundary value problem for the system of $l_1 + r_1$ ODEs.

One can also use the same approach to compute the stationary density when $p(x)$ satisfies [**E1**] and the distributions F_L, F_R are finite mixtures of $\beta(1, z)$ with different z-values.

Example 1. Suppose that the function p is polynomial: for a $k \in \mathbb{N}$, one has $p(x) = \sum_{n=0}^{k} p_n x^n = \sum_{n=0}^{k} q_n (x-1)^n$. Assuming that $p_0 < 1$ and $q_0 > 0$ to ensure that condition [**E1**] is satisfied, and that $F_L = \beta(1, l)$, $F_R = \beta(1, r)$, we see that X has stationary density of the form

$$\pi(x) = C x^{l(1-p_0)-1}(1-x)^{rq_0-1}(l + (r-l)x) \\ \times \exp\left(r \sum_{n=1}^{k} \frac{q_n}{n}(x-1)^n - l \sum_{n=1}^{k} \frac{p_n}{n} x^n \right), \quad x \in (0,1).$$

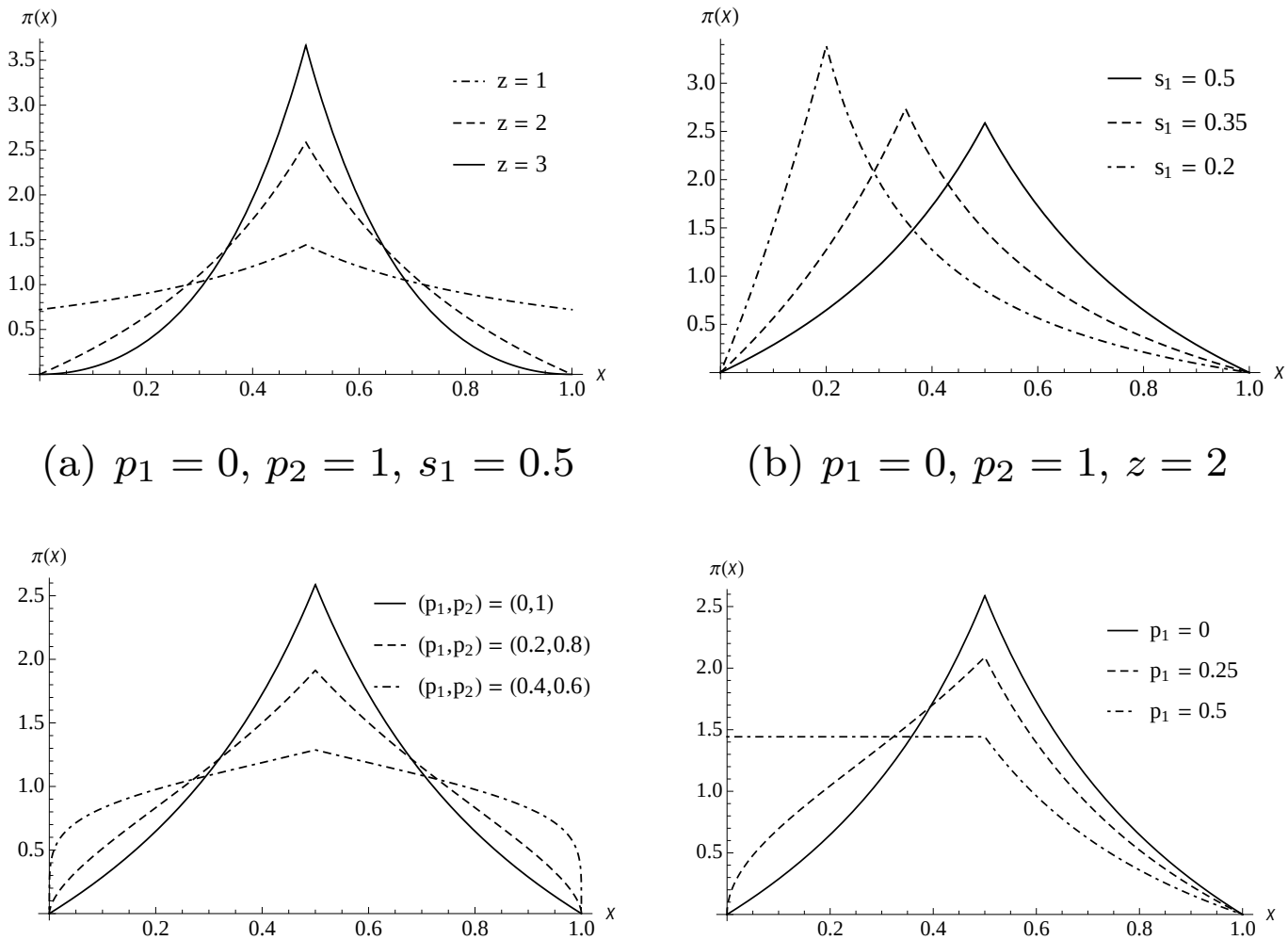
In particular, if $p(x)$ is linear: $p(x) = cx + (1-b)(1-x)$, where $b, c \in (0, 1]$, and $F_L = F_R = \beta(1, z)$, $z > 0$, we immediately obtain that X has stationary distribution $\beta(bz, cz)$. Even this extends a couple of known results.

Example 2. Piecewise constant $p(x)$, $F_L = F_R = \beta(1, z)$, $z > 0$. In that case one can obtain multimodal stationary densities, with modes located at the discontinuity points of $p(x)$.

Suppose that $p(x) = p_i \in [0, 1]$, $s_{i-1} \leq x < s_i$, $i = 1, \ldots, k$, where $k \in \mathbb{N}$, $0 = s_0 < s_1 < \cdots < s_k = 1$, and $p_1 < 1$, $p_k > 0$. Then $p(x)$ satisfies [**E1**], and so X has stationary density

$$\pi(x) = C_i x^{z(1-p_i)-1}(1-x)^{zp_i-1}, \quad s_{i-1} \leq x < s_i, \quad i = 1, \ldots, k,$$

where the constants $C_i > 0$ are easily computed using continuity of π. That is, the (continuous) stationary density of X is "glued" of pieces of different beta densities on disjoint intervals (s_{i-1}, s_i). This extends Theorem 1 in Bialkowski (2002) (giving a special case of the above general formula, corresponding to $(k, z, p_1, p_2, s_1) = (2, 1, 0, 1, 1/2)$).



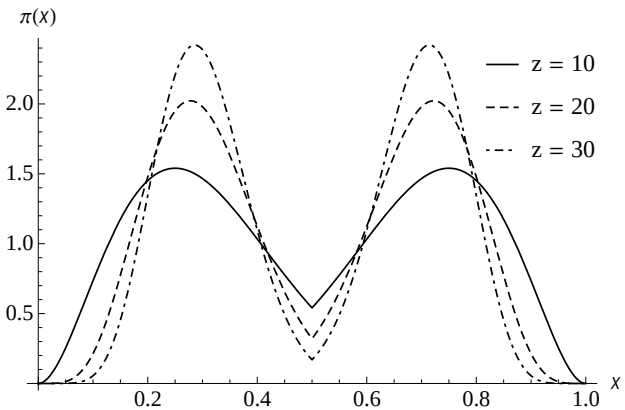
(a) $p_1 = 0,\ p_2 = 1,\ s_1 = 0.5$

(b) $p_1 = 0,\ p_2 = 1,\ z = 2$

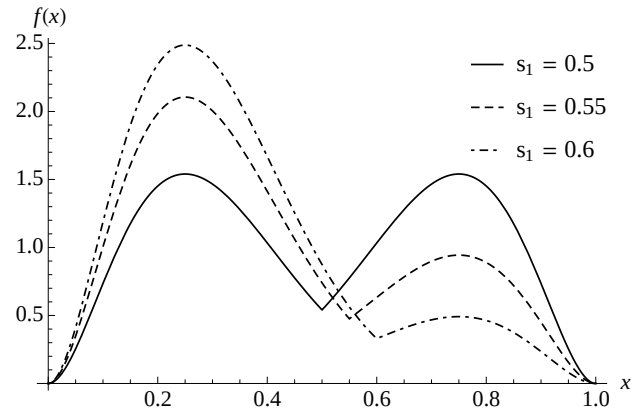
(c) $s_1 = 0.5,\ z = 2$

(d) $s_1 = 0.5,\ p_2 = 1,\ z = 2$

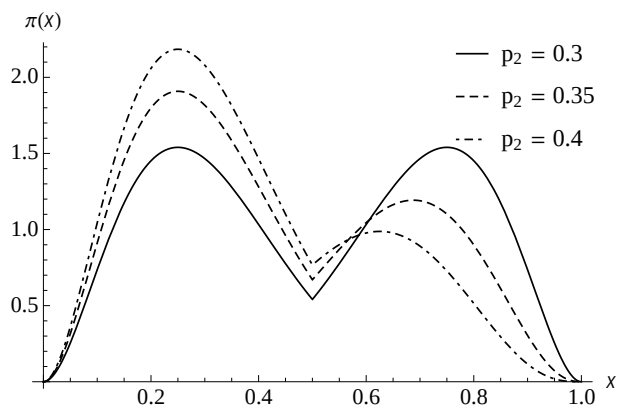
Figure 1: Some examples of peaked stationary densities generated by piecewise constant $p(x)$ with $k = 2$.



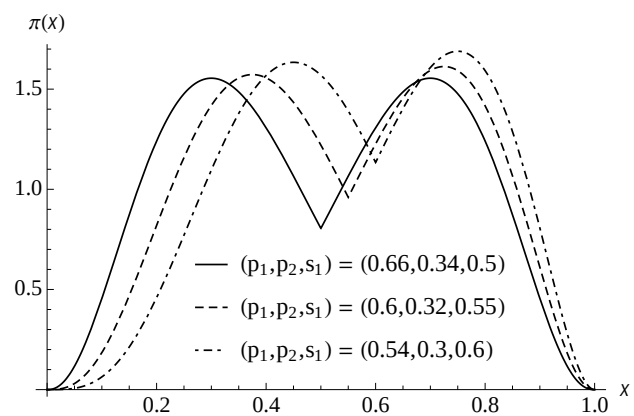
(a) $p_1 = 0.7$, $p_2 = 0.3$, $s_1 = 0.5$



(b) $p_1 = 0.7$, $p_2 = 0.3$, $z = 10$

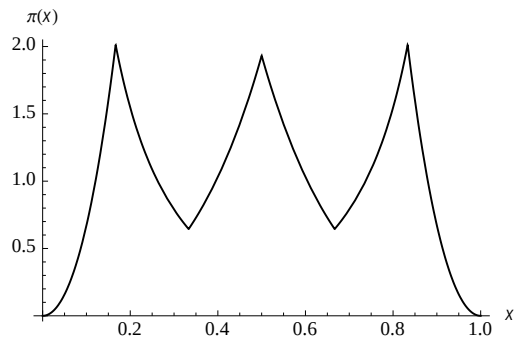


(c) $p_1 = 0.7$, $s_1 = 0.5$, $z = 10$

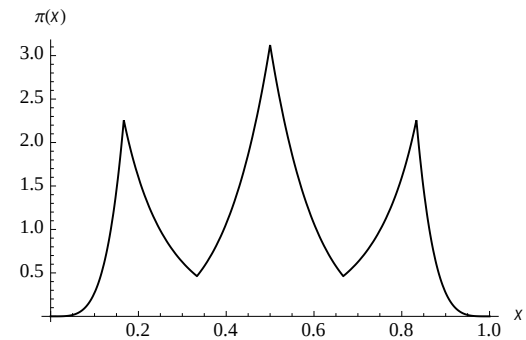


(d) $z = 10$

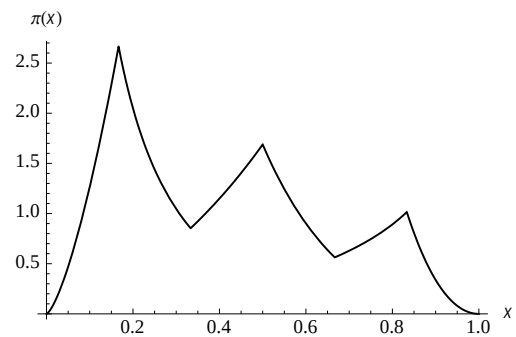
Figure 2: Some examples of bimodal stationary densities generated by piecewise constant $p(x)$ with $k = 2$.



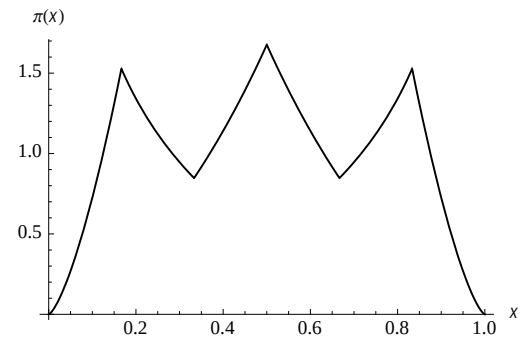
(a) $z = 3$, $\boldsymbol{p} = (0, 1, 0, 1, 0, 1)$



(b) $z = 5$, $\boldsymbol{p} = (0, 1, 0, 1, 0, 1)$



(c) $z = 3$, $\boldsymbol{p} = (1/5, 1, 1/5, 1, 1/5, 1)$



(d) $z = 3$, $\boldsymbol{p} = (1/5, 4/5, 1/5, 4/5, 1/5, 4/5)$

Figure 3: Some examples of trimodal stationary densities generated by piecewise constant $p(x)$, where $k = 6$, $\boldsymbol{p} := (p_1, \ldots, p_6)$, $s_1 = 1/6$, $s_2 = 1/3$, $s_3 = 1/2$, $s_4 = 2/3$, $s_5 = 5/6$.

EXAMPLE 3. A robot coverage algorithm and random search.

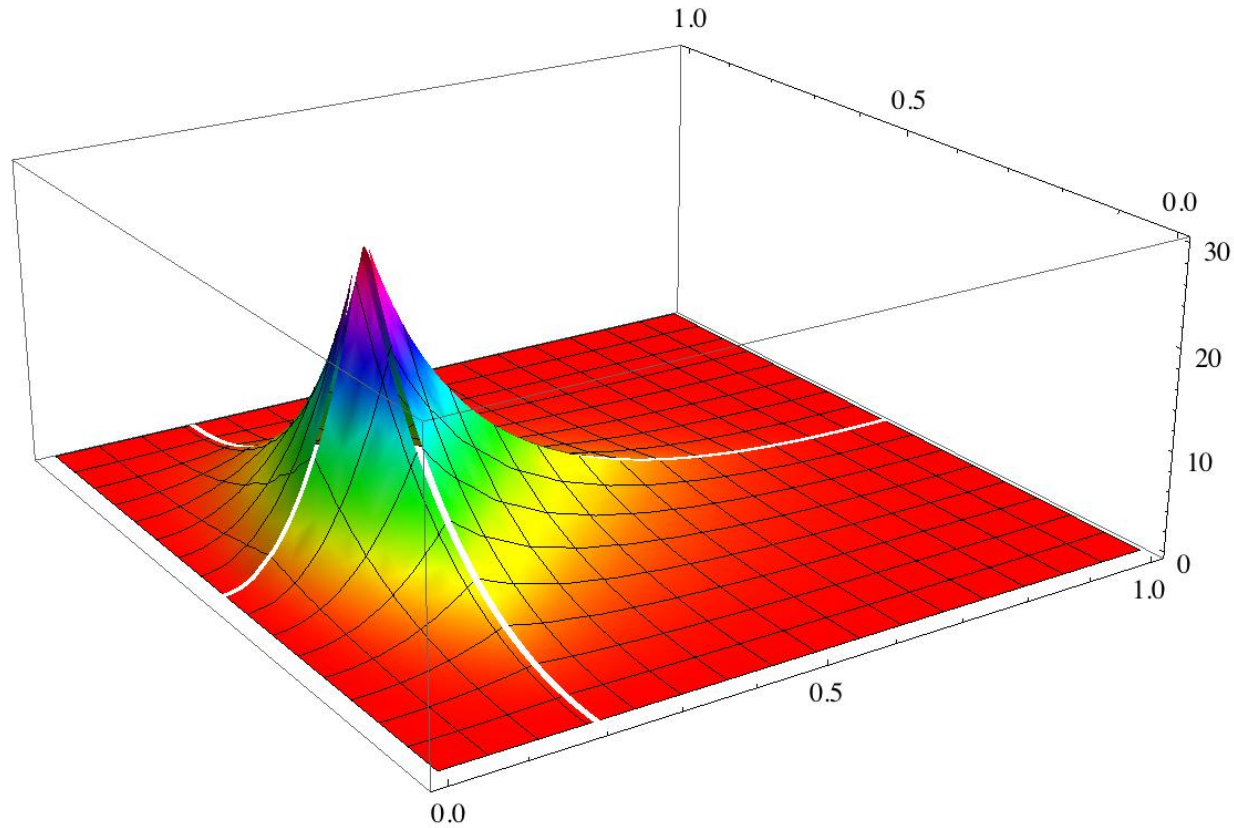


Figure 4: The stationary density of X for the robot converge algorithm: here $p_i = \mathbf{1}(x_i > y_i)$, with a "point of interest" at $(y_1, y_2) := (0.2d_1, 0.5d_2)$ (as in Ramli & Leng (2010)), $l_1 = r_1 = l_2 = r_2 = 3$, and $d_1 = d_2 = 1$.

Simulating random vectors with given densities on surfaces etc, using MCMC (KB (1994)).

Recall: if the transition density (w.r.t. measure μ) is symmetric: $f(x, y) = f(y, x)$, then μ is invariant for X.

Random directions algorithm (Turčin (1971), re-discovered by R.L. Smith (1984), sequential directions (Ritov (1989)). No good for surfaces.

Lalley & Robbins (1988): "princess and monster game" on a convex subset of $\mathbb{R}^2$. A beautiful idea! Extends.