Question 1 (2016 Q3)

The Integration by Parts formula is: $\int f \cdot g' \, \mathrm{d}x = f \cdot g - \int f' \cdot g \, \mathrm{d}x$

For definite integrals, it becomes: $\int_a^b f \cdot g' \, \mathrm{d}x = \left[f \cdot g \right]_a^b - \int_a^b f' \cdot g \, \mathrm{d}x$

You may use these Integration by Parts formulae in this question.

(a) A function $f(x)$, where x is a real number, is defined implicitly by the following formula:

$$f(x) = x - \int_0^{\frac{\pi}{2}} f(x) \sin x \, \mathrm{d}x$$

Find the explicit expression for $f(x)$ in its simplest form.

(b) A curve passing through the point $(1,1)$ has the property that, at each point P (x,y) on the curve, the gradient of the curve is $x - 2y$, that is, $\frac{\mathrm{d}y}{\mathrm{d}x} = x - 2y$.

(i) Show that $\frac{\mathrm{d}\left(\mathrm{e}^{2x} y\right)}{\mathrm{d}x} = x\mathrm{e}^{2x}$.

(ii) Hence, or otherwise, find the equation of the curve.

Question 2 (2016 Q4)

(a) P is a point on the right branch of the hyperbola $\frac{x^2}{4}-\frac{y^2}{36}=1$.

θ is the angle between the two asymptotes, l_1 and l_2, of the hyperbola, as shown on the diagram below.

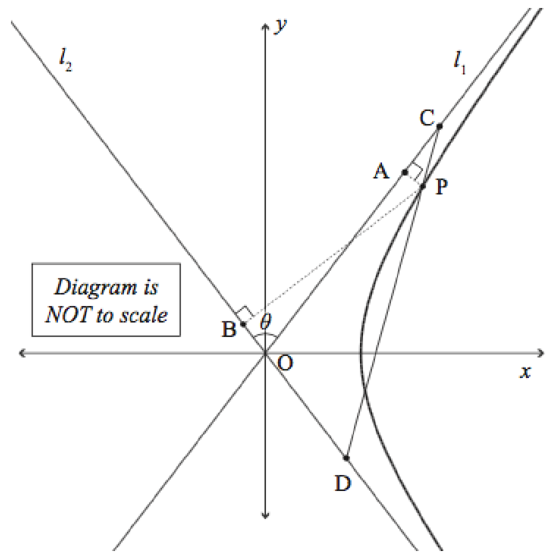
Show that $\sin\theta = 0.6$.

(b) From P, lines are drawn perpendicular to the asymptotes. The points where the perpendiculars meet the asymptotes are A and B, as shown on the diagram.

Show that the product of the lengths of the two segments PA and PB is constant, and state its value.

(c) Consider a case where P is located in the first quadrant. Through P, draw another line CD, where C is on line l_1, D is on line l_2, and P is between C and D, such that $CP : PD = 1 : \lambda$.

Find the value of λ, which will minimise the area of the triangle COD, and find this area.



Question 3 (2016 Q5)

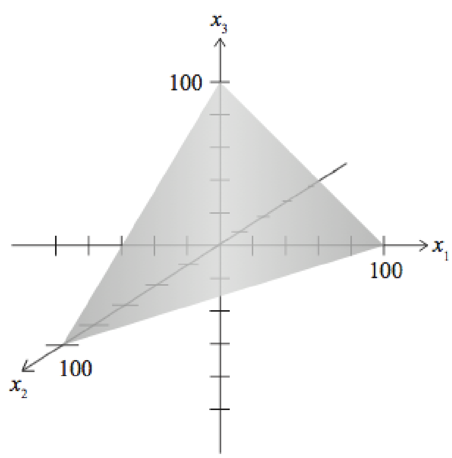
A student is to sit an examination. The questions are divided into three groups. The student may answer any question from any group so long as the total number of questions answered does not exceed 100. The groups are characterised as follows:

- Group 1 – easy, worth four marks each, and will take an average time of two minutes per question to answer.
- Group 2 – moderate difficulty, worth five marks each, and will take an average time of three minutes per question to answer.
- Group 3 – the most difficult, worth six marks each, and will take an average time of four minutes per question to answer.

The total time available to the student is 3 ½ hours. The questions in groups 1 and 2 are the most mechanical and the student can tolerate only 2 ½ hours of this kind of work before losing motivation.

One of the constraints is $x_1 + x_2 + x_3 \leq 100$, where x_1, x_2, and x_3 represent the number of questions answered from Groups 1, 2, and 3, respectively.

Graphically, this constraint can be represented as a plane in three dimensions. It defines a region of solutions to consider, lying between the plane and the origin, including the origin and the plane.



This graph is repeated on pages 26 and 27 of your answer booklet.

Write down the remaining constraints and the objective function.

What combination of questions should the student answer for a maximum grade, assuming all answered questions are correct?

Question 4 (2015 Q1)

(a) A solid of revolution is a three-dimensional figure formed by revolving a plane area around a given axis.

The surface area of a solid of revolution, which has been revolved 360° around the x-axis, is given by:

$$\text{surface of revolution} = \int_a^b \left(2\pi\, f(x)\sqrt{1+\left[f'(x)\right]^2} \right) dx$$

Find the area of the surface of revolution obtained when the graph of $f(x) = x^3 + \frac{1}{12x}$, from $x = 1$ to $x = 3$, is revolved 360° around the x-axis.

Question 5 (2013 Q6)

(a) By considering the expansion of $(\text{cis}\,\theta)^5$, or otherwise, prove both of the following identities.

$$\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$$

$$\sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$$

(b) Answer ONE of the following options.

EITHER

A teacher sets 99 homework questions for her Calculus class each week, of three different types: easy, difficult, and impossible. The number of questions of each type, given in week *n,* are represented by x_n, y_n, and z_n respectively.

The teacher uses the following **system of linear equations** to vary the number of questions of each type given each week.

$$x_{n+1} = 0.8x_n + 0.7y_n + 0.6z_n$$
$$y_{n+1} = 0.1x_n + 0.2y_n + 0.4z_n$$
$$z_{n+1} = 0.1x_n + 0.1y_n$$

Her class notice that the number of questions of each type stabilises after several weeks. That is, in the long run they notice that $x_{n+1} = x_n$, $y_{n+1} = y_n$, and $z_{n+1} = z_n$.

How many questions of each type will the teacher give each week once the numbers stabilise?

OR

Use your knowledge of **ellipses** to sketch all points in the complex plane satisfying the following inequalities, where k is a positive constant.

$$k \le |z + \text{i}| + |z - \text{i}| \le 2k$$

Question 6 (2015 Q2)

(a) Solve the following simultaneous equations for real values of x and y:

$$\begin{cases} 9^{2x+y} - 9^{x} \times 3^{y} = 6 \\ \log_{x+1}(y+3) + \log_{x+1}(y+x+4) = 3 \end{cases}$$

(b) A car is travelling at night along a road shaped like a parabola, with its vertex at the origin. The car starts at a point 100 m west and 100 m north of the origin. There is a statue located 100 m east and 50 m north of the origin.

At what point on the road will the car's headlights illuminate the statue?