Question One (2008 Q1)

(a) We are always looking for more efficient ways to store and stack things.
Cross sections tell us a lot about the stack and the space being filled.
Figure 1 shows the cross section of a hexagonal stack.

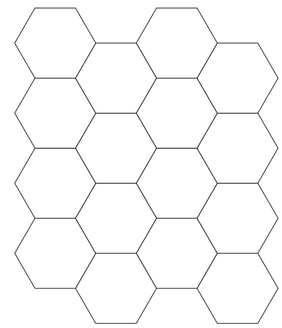

Figure 1

Show that the area of a regular hexagon with edge length s millimetres is $\frac{3\sqrt{3}}{2}s^2$ square millimetres.

Hence show that the total area of the hexagonal stack in Figure 1 is $24\sqrt{3}s^2$ square millimetres.

(b) Although a single cell of a bee's honeycomb has a hexagonal base, it is not a hexagonal prism. The complete cell more commonly has the shape shown in Figure 2.

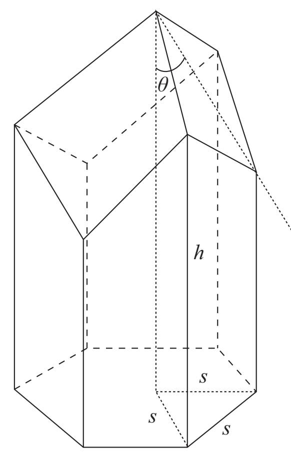

Figure 2

The surface area of this cell is given by

$$A = 6hs + \frac{3}{2}s^2\left(\frac{-\cos\theta}{\sin\theta} + \frac{\sqrt{3}}{\sin\theta}\right)$$

where h, s, θ are as shown in Figure 2.
Keeping h and s fixed, for what angle, θ, is the surface area a minimum?
You do not need to prove it is a minimum.

(c) Another cell, as described in part (b) above, has s not fixed, but increasing at a rate proportional to $\sin\theta$, where $0 < \theta < \frac{\pi}{2}$.

This rate is equal to $\sqrt{2}$ when $\theta = \frac{\pi}{4}$.

Keeping h fixed, at what rate is the surface area decreasing when θ is equal to the value found in part (b) above? Give your answer in surd form, in terms of h and s.

Question Two (2013 Q1)

Prince Rupert's drops are made by dripping molten glass into cold water. A typical drop is shown in Figure 1.

Figure 1: A seventeenth century drawing of a typical Prince Rupert's drop.
Image from *The Art of Glass* p 354, translated and expanded from
L'Arte Vetraria (1612) by Antonio Neri.

A mathematical model for a drop as a volume of revolution uses $y = \sqrt{\phi\left(e^{-x} - e^{-2x}\right)}$ for $x \geq 0$, and is is shown in Figure 2, where ϕ is the golden ratio $\phi = \dfrac{1+\sqrt{5}}{2}$.

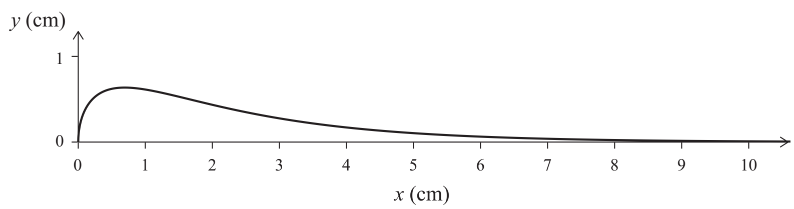

Figure 2: A mathematical model for a drop as a volume of revolution.

(a) Where is the modelled drop widest, and how wide is it there?

(b) The drop changes shape near B in Figure 1, where the concavity of the revolved function is zero.

Use $\dfrac{d^2y}{dx^2} = \sqrt{\phi}\,\dfrac{(e^{2x} - 6e^{x} + 4)}{y^2 e^{4x}}$ to find the exact x coordinate of B.

Question Three (2013 Q2)

1 The **inner product** of two continuous functions f and g over the interval $a \le x \le b$ is

$$\langle f, g \rangle_a^b = \int_a^b f(x)g(x)\,\mathrm{d}x$$

2 The **norm** of the continuous function f over the interval $a \le x \le b$ is

$$\|f\|_a^b = \sqrt{\langle f, f \rangle_a^b}$$

3 The **angle θ between two functions** f and g over the interval $a \le x \le b$ is given by

$$\cos\theta = \frac{\langle f, g \rangle_a^b}{\|f\|_a^b \cdot \|g\|_a^b} \; \textit{where} \; \|f\|_a^b \ne 0, \; \|g\|_a^b \ne 0$$

4 Two functions f and g are **orthogonal** over the interval $a \le x \le b$ if the angle between them is $\frac{\pi}{2}$

(a) Find the exact values of k for which $f(x) = kx + 1$ and $g(x) = x + k$ are orthogonal over $0 \le x \le 1$.

(b) Consider the functions $p(x) = 3x - 4$ and $q(x) = 9x - 5$ over $0 \le x \le 1$.

Find the exact angle between the two functions.

(c) For what positive integers n and m are $\sin(nx)$ and $\sin(mx)$ orthogonal over $0 \le x \le 2\pi$?

Question Four (2013 Q3)

(a) A function f is **even** if $f(-x) = f(x)$ for all x in its domain.
A function f is **odd** if $f(-x) = -f(x)$ for all x in its domain.

(i) Recall that a polynomial is a function in the form $p(x) = a_0x^0 + a_1x^1 + \ldots + a_nx^n$.

Describe which polynomials are even, and which are odd, and which are neither.

(ii) Suppose that g is any even differentiable function defined for all real numbers (not necessarily a polynomial).

Use the limit definition of the derivative to prove that $\frac{\mathrm{d}g}{\mathrm{d}x}$ is an odd function.

(b) Suppose $y = \mathrm{e}^{-x}\sin(kx)$, where k is a non-zero constant.

Find the values of k for which $\frac{\mathrm{d}^3y}{\mathrm{d}x^3} = Cy$, and hence find the value of C.

Question Five (2013 Q4)

(a) Find all the points which satisfy $z^n = z$, where z is a complex number, and n is a whole number where $2 \leq n \leq 9$.

How many different solutions are there altogether?

(b) (i) The relativistic rocket equation is below.

$$\frac{m_0}{m_1} = \left(\frac{1 + \frac{\Delta v}{c}}{1 - \frac{\Delta v}{c}} \right)^{\frac{c}{2u}}$$

Show that this equation rearranges to $\Delta v = c \cdot \tanh\left(\frac{u}{c} \ln\left(\frac{m_0}{m_1} \right) \right)$

where the hyperbolic tangent function is $\tanh(x) = \frac{e^{2x} - 1}{e^{2x} + 1}$.

(ii) The relativistic rocket equation is derived from the following differential equation, where u and c are constants.

$$\frac{dM}{dv} = \frac{-M}{u\left(1 - \frac{v^2}{c^2}\right)}$$

Show that $\ln M = \frac{-c}{2u} \ln\left(\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}} \right)$ is a solution of this differential equation.

Question Six (2012 Q3)

(a) (i) Find $\frac{\mathrm{d}}{\mathrm{d}x}\big(x\cos(x)\big)$ and use this result to find $\int x\sin(x)\,\mathrm{d}x$.

(ii) Hence find the value of $\int_0^{n\pi} x\sin(x)\,\mathrm{d}x$ for positive integer values of n.

(b) Consider the points in the region R shown in the Argand diagram of Figure 2, consisting of all points in a right-angled sector of radius 1, except for the point $z = 0.8\,\text{cis}\frac{\pi}{6}$.

Sketch the region containing all points w^3, where w is a point within the region R.

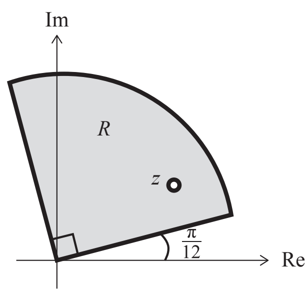

Figure 2: The region R in an Argand diagram (note the point z is not a point in R).

Question Seven (2012 Q4)

(a) Consider the function $f(x) = \log_m x + \log_x m$ defined for $x > 1$.

For a fixed value of $m > 1$, find the minimum value of $f(x)$.

Clearly explain the steps of your working.

(b) The following equation relates two real variables x and y, where q is a fixed constant.

$$y^4 + (1 - q^2)x^2y^2 - q^2x^4 + q^2x^2 - y^2 = 0$$

Sketch all points that satisfy the equation.

You might start by substituting $y^2 = q^2x^2$ and interpreting the result.

(c) Prove the following trigonometric identity:

$$2\tan(2x) \cdot \big(\tan(x) - 1\big)^2 \big(\tan(x) + 1\big)^2 = \tan(4x) \cdot \big(\tan^2(x) - 2\tan(x) - 1\big)\big(\tan^2(x) + 2\tan(x) - 1\big)$$

Note that you do not need to work in terms of $\sin(x)$ and $\cos(x)$ to prove this identity.

Question Eight (2011 Q2)

(a) A circular pond is 5 metres in radius. The volume (in cubic **metres**) of water in the pond when the water is x metres from the top is $V(x) = \frac{250\pi}{3} - 25\pi x + \frac{\pi}{3}x^3$.

Rain falls at the rate of 15 **millimetres** per hour.

How fast is the depth of water in the pond rising when it is 3 metres from the top?

(b) Figure 1 below shows the function $g(x) = \sqrt{1-\sqrt{x}}$ as a solid line.

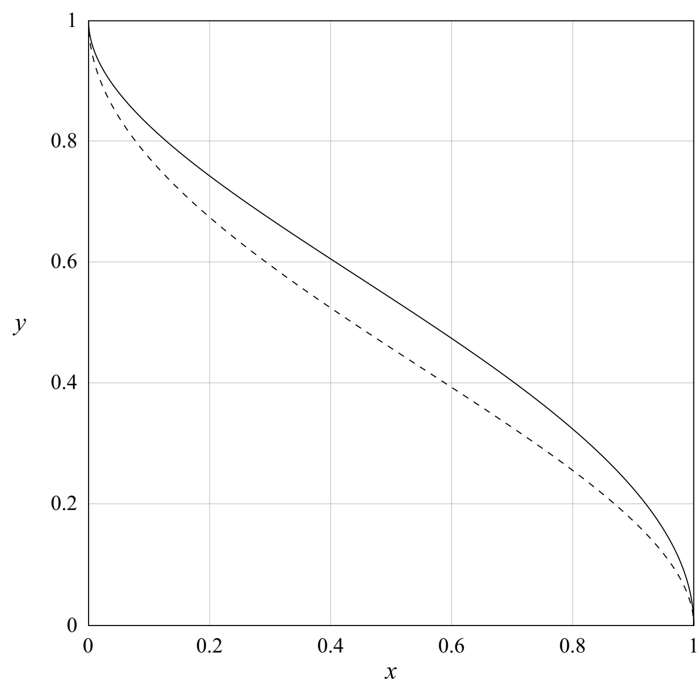

Figure 1: Graph of $y = g(x)$, and the curve rotated about the point $\left(\frac{1}{2}, \frac{1}{2}\right)$.

(i) Use differentiation to show that $\int g(x)\,\mathrm{d}x = A\left(1-\sqrt{x}\right)^{1.5}\left(2+3\sqrt{x}\right)+C$, and find the value of A.