Two locality properties in two dimensions

Jesse Goodman

Department of Statistics
University of Auckland

4 December 2017

Two locality properties in two dimensions
. . . one locality property per dimension?

Jesse Goodman

Department of Statistics
University of Auckland

4 December 2017

Outline

1. Some random discrete paths
   - Example: loop-erased random walk
   - Domain Markov property
   - Example: percolation exploration process

2. SLE_κ processes and the classical locality property
   - The idea of scaling limits
   - Example: scaling limit of percolation exploration process
   - The Riemann mapping theorem; or, why $\mathbb{C}$ is better than $\mathbb{R}^2$
   - Schramm-Loewner evolution
   - The classical locality property for SLE_6

3. The enclosing locality property
   - What it says
   - Idea of the proof

Outline

1. Some random discrete paths
   - Example: loop-erased random walk
   - Domain Markov property
   - Example: percolation exploration process
2. SLE_κ processes and the classical locality property
   - The idea of scaling limits
   - Example: scaling limit of percolation exploration process
   - The Riemann mapping theorem; or, why $\mathbb{C}$ is better than $\mathbb{R}^2$
   - Schramm-Loewner evolution
   - The classical locality property for SLE_6
3. The enclosing locality property
   - What it says
   - Idea of the proof

Example: loop-erased random walk

Consider a discrete connected domain $D \subset \mathbb{Z}^2$ in the square lattice with specified boundary points a, b. Run a simple symmetric random walk in $\mathbb{Z}^2$ started at a and conditioned to avoid $\partial D \setminus \{b\}$ until it is stopped at b.

Definition

Loop-erased random walk from a to b in D is the random simple path from a to b formed by sequentially erasing loops in the random walk.

Write $\gamma^{\mathrm{LERW}}_{a,b,D}(n)$ for the path of the loop-erased random walk.

Domain Markov property

Clearly the position of loop-erased random walk $\gamma^{\mathrm{LERW}}_{a,b,D}(n) \in \overline{D}$ will not be a Markov chain – this follows from the fact that it must be self-avoiding. However, a different object will have a Markov property.

How to condition on $\gamma^{\mathrm{LERW}}[0, n]$

The event that the initial segment $\gamma^{\mathrm{LERW}}[0, n]$ follows a specified initial segment $\gamma[0, n]$ means that:

- in the past, the random walk acted consistently with $\gamma[0, n]$;
  (i.e., the random walk travelled along the sequence $\gamma(0), \gamma(1), \ldots, \gamma(n)$, with the step to $\gamma(i)$ followed by zero or more loops that return to $\gamma(i)$ without visiting $\gamma[0, i-1]$, for each $1 \le i \le n$)
- in the future, the random walk avoids $\gamma[0, n]$

Domain Markov property

The event $\{\gamma^{\text{LERW}}[0,n] = \gamma[0,n]\}$ means that:

- in the future, the random walk avoids $\gamma[0,n]$

The requirement to avoid $\gamma[0,n]$ can be added to the requirement to avoid $\partial D \setminus \{b\}$.

Domain Markov property

The future evolution of $\gamma^{\text{LERW}}[n,\infty]$ is the same as LERW in $D_n = D \setminus \gamma^{\text{LERW}}[0,n]$ started from $\gamma^{\text{LERW}}(n)$.

i.e.,

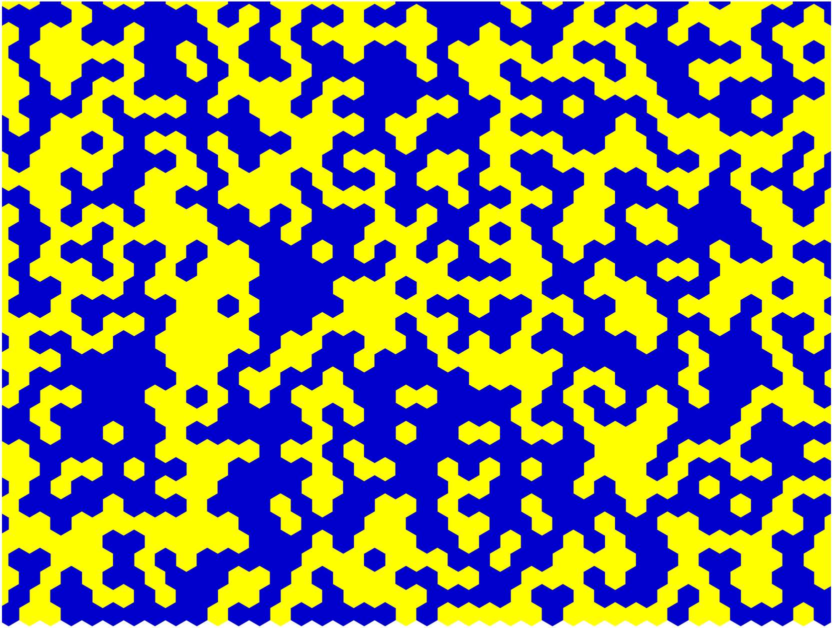
The pair $(\gamma^{\text{LERW}}(n), D_n)$ has the Markov property and, given $\gamma^{\text{LERW}}[0,n]$, the conditional law of $\gamma^{\text{LERW}}(n+\cdot)$ is the law of $\gamma^{\text{LERW}}_{\gamma(n),b,D_n}(\cdot)$.

Example: percolation exploration process

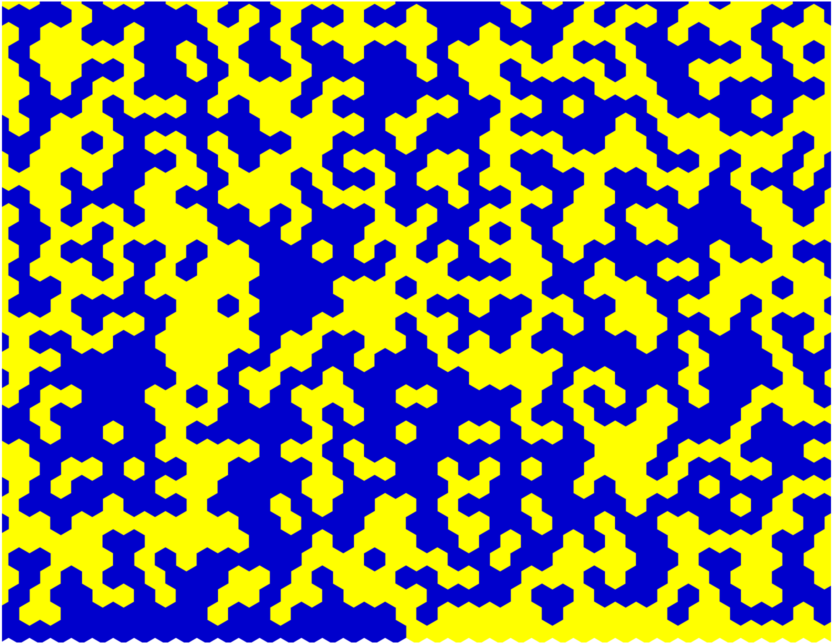
Percolation exploration process

Perform critical site percolation on the upper half-plane of the triangular lattice, and set the boundary sites to be open on one side of the starting point and closed on the other. Explore the interface between the open and closed cluster by following a path that always keeps open sites on one side and closed sites on the other.

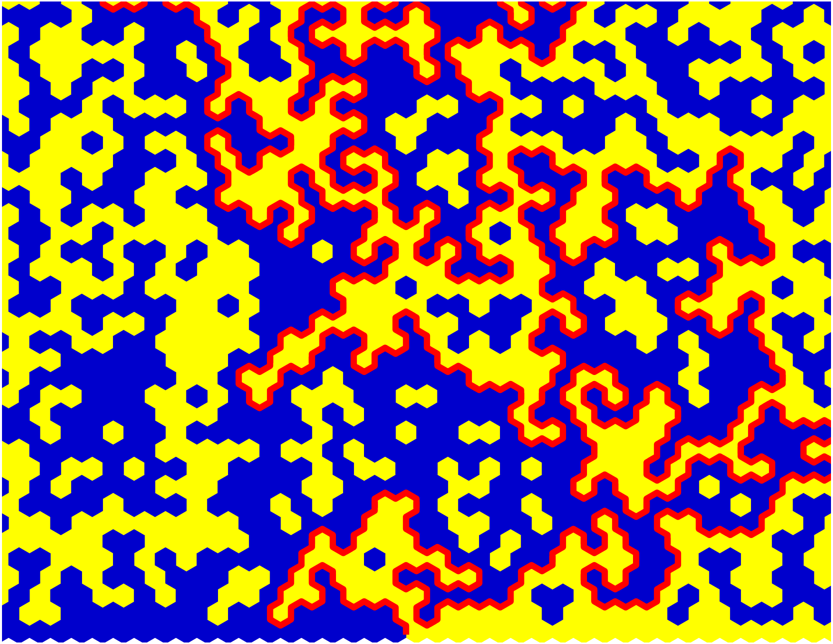
Picture: triangular lattice



Picture: triangular lattice



Picture: triangular lattice



Outline

1. Some random discrete paths
   - Example: loop-erased random walk
   - Domain Markov property
   - Example: percolation exploration process

2. SLE_κ processes and the classical locality property
   - The idea of scaling limits
   - Example: scaling limit of percolation exploration process
   - The Riemann mapping theorem; or, why $\mathbb{C}$ is better than $\mathbb{R}^2$
   - Schramm-Loewner evolution
   - The classical locality property for SLE_6

3. The enclosing locality property
   - What it says
   - Idea of the proof

The idea of scaling limits

Long version

Given a domain D, boundary points $a, b \in \partial D$, and a parameter $\delta > 0$:

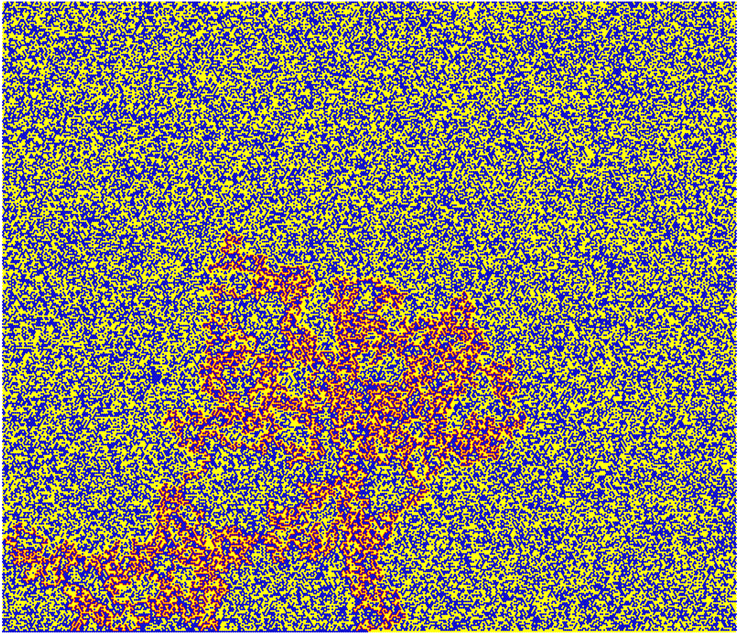
- Lay down a grid of spacing δ.
- Take D^δ to be the set of grid points lying inside D and a^δ, b^δ to be the closest grid points to a, b.
- Run the discrete process in D^δ using the grid of spacing δ.
- Take $\delta \to 0$.

We can then hope that the discrete path converges (in distribution, in the space of paths modulo time reparametrisation) to a limiting continuous path. The continuous path is the scaling limit of the discrete path.

Short version

Just zoom out.

Picture: scaling limit of percolation exploration process



Example: scaling limit of percolation exploration process

Percolation exploration process

Perform critical site percolation on the upper half-plane of the triangular lattice, and set the boundary sites to be open on one side of the starting point and closed on the other. Explore the interface between the open and closed cluster by following a path that always keeps open sites on one side and closed sites on the other.

Theorem

The scaling limit of the percolation exploration process (as the lattice spacing tends to 0) is (chordal) SLE_6 *in* $\mathbb{H}$ *from 0 to* ∞.

The Riemann mapping theorem;

or, why $\mathbb{C}$ is better than $\mathbb{R}^2$

Theorem

If D, D' are two (simply connected) domains in $\mathbb{C}$, then there is a one-to-one, onto, analytic function $g : D \to D'$.

- The function g is called a conformal mapping.
- If $a, b \in \partial D$ and $a', b' \in \partial D'$ are specified boundary points, it is possible to choose g so that $g(a) = a', g(b) = b'$.
- If $\gamma = \gamma(t)$ is a path in D, then $g(\gamma) = g(\gamma(t))$ is a path in D'.
- Conformal invariance means that if γ, γ' are random paths in D, D' then γ' has the same distribution as $g(\gamma)$.

An insight from theoretical physics

For various reasons, many (but not all) two-dimensional discrete models would be expected to have conformally invariant scaling limits.

The SLE_κ processes

The Schramm-Loewner evolution SLE_κ, for $\kappa \geq 0$, is a parametrised family of random paths $\gamma(t) = \gamma_{a,b,D}(t) \colon [0, \infty] \to \overline{D}$ joining a to b in the closure $\overline{D}$ of a domain $D \subset \mathbb{C}$. We normally consider the path γ to be defined modulo reparametrisation of time.

The SLE_κ processes for $\kappa \geq 0$

- have continuous non-self-crossing paths;
- are conformally invariant;
- have a suitable left-right symmetry; and
- satisfy the Domain Markov Property.

Moreover the SLE_κ processes are the only processes with these properties.

Loewner evolution

Consider now a domain $D \subset \mathbb{C}$ with specified boundary points a, b.

Theorem (Loewner evolution – chordal case)

Continuous non-self-crossing paths $\gamma(t) : [0, \infty] \to \overline{D}$ from $\gamma(0) = a$ to $\gamma(\infty) = b$, when parametrised appropriately, can be described in terms of a continuous driving function $U(t) : [0, \infty) \to \mathbb{R}$. In the case $D = \mathbb{H}$ and $b = \infty$, let D_t be the unbounded component of $\mathbb{H} \setminus \gamma[0, t]$ and let $g_t(z) : D_t \to \mathbb{H}$ be the unique conformal transformation satisfying $g_t(\infty) = \infty$ and $g_t(z) - z \to 0$ as $z \to \infty$. Then $g_t(z)$, and hence D_t and $\gamma(t)$, is uniquely determined by the Loewner evolution

$$\frac{\partial}{\partial t} g_t(z) = \frac{2}{g_t(z) - U(t)}.$$

Schramm-Loewner evolution

We want to study processes $\gamma_{a,b,D}(t)$ that

- have continuous non-self-crossing paths;
- have a suitable left-right symmetry; and
- are conformally invariant;
- satisfy the Domain Markov Property.

Interpreted in terms of Loewner evolutions, the driving function $U(t)$ should

- be continuous;
- be symmetric;
- have stationary independent increments.

Schramm-Loewner evolution

Definition

The (chordal) Schramm-Loewner evolution SLE_κ (from 0 to ∞ in $\mathbb{H}$), $\kappa \geq 0$, is the random path $\gamma(t)$ obtained from by setting $U(t) = \sqrt{\kappa} B(t)$ where B is a standard Brownian motion:

$$\frac{\partial}{\partial t} g_t(z) = \frac{2}{g_t(z) - \kappa B(t)}$$

- The scaling limit of loop-erased random walk is SLE_2
- The scaling limit of self-avoiding walk is $\mathrm{SLE}_{8/3}$ (conjecturally)
- The scaling limit of harmonic explorer is SLE_4
- The scaling limit of the percolation exploration process is SLE_6.
- The scaling limit of the uniform spanning tree curve is SLE_8

Locality

Consider two nested domains $D \subset D' \subset \mathbb{C}$ with $\{a, b\} \cap \overline{D' \setminus D} = \varnothing$.
Define the exit times from the smaller domain:

$$\tau = \inf\left\{t \colon \gamma_{a,b,D}(t) \notin \overline{D}\right\}, \qquad \tau' = \inf\left\{t \colon \gamma_{a,b,D'}(t) \notin \overline{D}\right\}.$$

Locality property

The locality property is satisfied if $\gamma_{a,b,D}[0,\tau]$ and $\gamma_{a,b,D'}[0,\tau']$ have the same law modulo a random time reparametrisation.

Theorem

SLE_κ *satisfies the locality property if and only if* $\kappa = 6$.

Outline

1. Some random discrete paths
   - Example: loop-erased random walk
   - Domain Markov property
   - Example: percolation exploration process
2. SLE_κ processes and the classical locality property
   - The idea of scaling limits
   - Example: scaling limit of percolation exploration process
   - The Riemann mapping theorem; or, why $\mathbb{C}$ is better than $\mathbb{R}^2$
   - Schramm-Loewner evolution
   - The classical locality property for SLE_6
3. The enclosing locality property
   - What it says
   - Idea of the proof

Enclosing locality

For an SLE curve $\gamma(t)$ in D, write D_t for the component of $D \setminus \gamma[0, t]$ containing b.
Given $A \subset \overline{D} \setminus \{b\}$ compact and containing at least two points, define the enclosing time

$$T_A = \inf\{t\colon \gamma[0, t] \text{ disconnects } A \text{ from } b \text{ inside } D\} = \inf\{t\colon A \cap D_t = \varnothing\}.$$

We want to know how D_{T_A} depends on the choice of domain D and starting point a.

Enclosing locality

$$T_A = \inf\{t\colon \gamma[0,t] \text{ disconnects } A \text{ from } b \text{ inside } D\} = \inf\{t\colon A \cap D_t = \varnothing\}.$$

Theorem

Let D^* be a domain with a specified boundary point $b \in \partial D_0$, and let A be a compact subset of $\overline{D^*} \setminus \{b\}$ such that $A \cap \partial D^*$ is connected and non-empty. Then the law of D_{T_A} is the same for any choice of domain D satisfying $D^* \setminus A \subset D$ and $\partial D^* \setminus A \subset \partial D$, and for any choice of $a \in \partial D$ with $a \notin \partial D^* \setminus A$.

Idea of the proof

Discrete approximation

The percolation exploration process also satisfies the corresponding discrete enclosing locality property: the discrete domain D_{T_A} is that domain bounded by the smallest monochromatic circuit that disconnects A from b, which only depends on sites in the common part of the various domains.

A key point in the proof is managing the passage to the scaling limit.
The discrete enclosing time does converge to a limiting enclosing time, but it is necessary to rule out that small gaps could vanish in the limit and produce an earlier limiting enclosing time.
The fact that this does not happen is related to the polychromatic 6-arm exponent and the half-plane 3-arm exponent.

Thank you.