# The distribution of the minimum of a sample from a distribution on $[0, \infty)$

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We give expansions for the distribution, density and moments of the sample minimum when sampling from a distribution on $[0, \infty)$ that is nearly analytic at 0.
When these distributions are analytic at 0, the expansions are in inverse powers of the sample size $n$. If not, they require a double expansion.

Suppose that we have a random sample of size $n$ with minimum $m_n$ from a distribution $F(x)$ on $[0, \infty)$. In §2 we suppose that $F(x)$ is analytic at $x_1$, and give expansions in powers of $n^{-1}$ for the distribution and moments of $m_n$.
In §3 we suppose that for some $a > 0, F(x)/x^a$ is analytic at 0: we give expansions in powers of $n^{-1/a}$ for the distribution, density and moments of the sample minimum $m_n$: but the coefficient of $n^{-i/a}$ is now a polynomial of degree $i$ in $n^{1/a-1}$.

# Expansions for analytic distributions

Suppose that $F(x) = P(X \leq x)$ on $[0, \infty)$ is analytic at 0. So

$$F(x) = \sum_{i=1}^{\infty} x^i g_i \text{ where } g_1 > 0. \tag{1}$$

We shall see that

$$Y_n = -g_1 n m_n \xrightarrow{\mathcal{L}} Y \text{ as } n \to \infty \text{ where} \tag{2}$$
$$P(Y \leq y) = e^y \text{ on } (-\infty, 0].$$

So $-Y$ is a standard exponential random variable and

$$E\ (-Y_n)^t \to E\ (-Y)^t = \Gamma(1+t) \text{ for } Re(t) > -1. \tag{3}$$

## Example

For the uniform on $[0, 1]$,
$F(x) = x,\ g_1 = 1,\ g_i = 0$ for $i \geq 2$. So $Y_n = -nm_n$. □

For $y \le 0$ set

$$z = -y/g_1, \; c_{i-1} = -\sum_{k=1}^{i} \tilde{B}_{ik}(g)/k, \tag{4}$$

$$t_j = t_j(z) = \sum_{k=0}^{j} \tilde{B}_{jk}(c) z^k / k!. \tag{5}$$

So $c_0 = -g_1, \; c_1 = -g_2 - g_1^2/2, \; c_2 = -g_3 - g_1 g_2 - g_1^3/3,$

$$\begin{aligned} c_3 &= -g_4 - g_1 g_3 - g_2^2/2 - g_1^2 g_2 - g_1^4/4, \\ c_4 &= -g_5 - g_1 g_4 - g_2 g_3 - g_1^2 g_3 + g_1^3 g_2 - g_1^5/5, \end{aligned} \tag{6}$$

$t_0 = 1, \; t_1 = z c_1, \; t_2 = z c_2 + z^2 c_1^2/2,$

$t_3 = z c_3 + z^2 c_1 c_2 + z^3 c_1^3/6,$

$t_4 = z c_4 + z^2 (c_1 c_3 + c_2^2/2) + z^3 c_1^2 c_2 + z^4 c_1^4/24,$

$t_5 = z c_5 + z^2 (c_1 c_4 + c_2 c_3) + z^3 (c_1^2 c_3/2 + c_1 c_2^2/2) + z^4 c_1^3 c_2/6 + z^5 c_1^5/12$

**Theorem**

*For $y \le 0$,*

$$P_n(y) = P(Y_n \le y) = e^y \sum_{j=0}^{\infty} (z/n)^j \; t_j(z). \tag{7}$$

From this expansion for the distribution of $m_n$, it's easy to get expansions for its density and moments.

PROOF Set

$$S = \sum_{i=1}^{\infty} (z/n)^i f_i, \; P_n = P_n(y) = F(-z/n)^n = (1+S)^n,$$

$$T_0 = \sum_{j=1}^{\infty} (z/n)^j c_j.$$

Then

$$T_0^k = \sum_{j=k}^{\infty} \tilde{B}_{jk}(c)\;(z/n)^j, \; S^k = \sum_{i=k}^{\infty} \tilde{B}_{ik}(f)\;(z/n)^i,$$

$$\ln P_n = n \ln(1+S) = n \sum_{k=1}^{\infty} (-1)^{k-1} S^k / k = n \sum_{i=1}^{\infty} (z/n)^i c_{i-1}$$

$$= y + zT_0,$$

$$P_n = e^y \sum_{k=0}^{\infty} (zT_0)^k / k! = e^y \sum_{j=0}^{\infty} (z/n)^j t_j.$$

## Example

For the uniform on $[0,1]$, $z=-y$,

$$
\begin{aligned}
c_r &= -1/(r+1),\\
t_0(z) &= 1,\ \ t_1(z) = -z2,\ \ t_2(z) = -z/3 + z^2/8,\\
t_3(z) &= -z/4 + z^2/6 - z^3/48,\\
t_4(z) &= -z/5 + 13z^2/72 - z^3/12 + z^4/384,\\
t_5(z) &= -z/6 + 11z^2/60 - 17z^3/288 + z^4/144 - z^5/3840.
\end{aligned}
$$

# Expansions for nearly analytic distributions

Suppose that $F(x)$ is a distribution on $[0, \infty)$ with

$$F(x) = x^{a-1} \sum_{i=1}^{\infty} x^i g_i \text{ where } a > 0, g_1 > 0. \tag{8}$$

For $y \geq 0$, set $\gamma = a - 1,\ N = (ng_1)^{1/a},\ y_N = y/N,\ Y_n = Nm_n,$ (9)

$$G(y) = \exp(-y^a),\ C_j(x) = C_j = \sum_{k=1}^{j} \tilde{B}_{jk}(g)\ x^{k\gamma}/k,$$

$$d_i(x) = d_i = xC_{i+1}(x),$$

$$E_i = N^a y^i d_i(y_N) = \sum_{j=0}^{i} E_{ij}\ \delta^j \text{ where } \delta = N^{1-a} = (ng_1)^{1/a-1},$$

(10)

$$E_{ij} = b_{ij} y^{(j+1)a+i-j}, \ b_{ij} = \tilde{B}_{i+a,j+1}(g)/(j+1) : E_0 = g_1 y^a,$$
$$E_{i0} = g_{i+1} y^{a+i}, \ E_{i,i-1} = g_1^{i-1} g_2 y^{ia+1}, \ E_{ii} = g_1^{i+1} y^{ia+a}/(i+1),$$
$$E_{31} = (g_1 g_3 + g_2^2/2) y^{2a+2}, \ E_{41} = (g_1 g_4 + g_2 g_3) y^{2a+3},$$
$$E_{42} = (g_1^2 g_3 + g_1 g_2^2) y^{3a+2}.$$

Set $h_k = (-g_1)^{-k}/k!$. Then

$P(Y_n > y) = G(y)Q_n$ where

$$Q_n = \sum_{i=0}^{\infty} N^{-i}\ P_{ni},\ \ P_{ni} = \sum_{k=0}^{i} \tilde{B}_{ik}(E)\ h_k, \tag{11}$$

$$P_{n0} = 1,\ P_{n1} = E_1 h_1,\ P_{n2} = E_2 h_1 + E_1^2 h_2,$$

$$P_{n3} = E_3 h_1 + 2E_1 E_2 h_2 + E_1^3 h_3,$$

$$P_{n4} = E_4 h_1 + (2E_1 E_3 + E_2^2)h_2 + 3E_1^2 E_2 h_3 + E_1^4 h_4,$$

$$E_1 = g_2 y^{a+1} + g_1^2 y^{2a}\delta/2,$$

$$E_2 = g_3 y^{a+1} + g_1 g_2 y^{2a}\delta + g_1^3 y^{3a}\delta^2/3,$$

$$E_3 = g_4 y^{a+1} + (g_1 g_3 + g_2^2/2)y^{2a+2}\delta + g_1^2 g_2 y^{3a+1}\delta^2 + g_1^4 y^{4a}\delta^3/4.$$

Also

$$\tilde{B}_{ik}(E) = \sum_{j=0}^{i} B_{ikj}(y)\ \delta^j \text{ where } B_{ikj}(y) = b_{ikj}\ y^{(j+k)a+i-j}, \tag{12}$$

$$b_{ik0} = \tilde{B}_{ik}(e),\ \ e_i = g_{i+1},\ \ b_{i10} = g_{i+1},\ \ b_{ii0} = g_2^i,$$

$$b_{i1j} = \text{ coefficient of } \delta^j \text{ in } E_i \text{ above}:$$

$$b_{i1,i-1} = g_{i-1}g_2,\ \ b_{i1i} = g_1^{i+1}/(i+1), \tag{13}$$

$$b_{ii1} = ig_1^2 g_2^{i-1}/2, b_{iii} = (g_1^2/2)^i, b_{iij} = \binom{i}{j}(g_1^2/2)^j g_2^{i-j}, \tag{14}$$

$$b_{311} = g_1 g_3 + g_2^2/2,\ \ b_{320} = 2g_2 g_3,\ \ b_{321} = 2g_1 g_2^2 + g_1^2 g_3,$$

$$b_{322} = 5g_1^3 g_2/3,\ \ b_{323} = g_1^5/3.$$

So $P_{n0} = 1$ and for $i \geq 1, P_{ni}$ of (11) can be written

$$P_{ni} = \sum_{j=0}^{i} P_{ij}(y)\ \delta^j \text{ where } P_{ij}(y) = \sum_{k=1}^{i} B_{ikj}(y)\ h_k. \qquad (15)$$

So these $b_{ikj}$ with (12) are enough to give $P_{ni}$ for $i = 1, 2, 3$.

$P_{n4}$ needs

$$b_{411} = g_1 g_4 + g_2 g_3,\ b_{421} = g_1^2 g_4/2 + 3g_1 g_2 g_3 + g_2^3/3,$$
$$b_{431} = 3g_1^2 g_2 g_3 + 3g_1 g_2^3,\ b_{412} = g_1^2 g_3 + g_1 g_2^2,$$
$$b_{422} = 7g_1^2 g_2^2/2 + 5g_1^3 g_3/3,\ b_{432} = 3g_1^4 g_3/4 + 4g_1^3 g_2^2,$$
$$b_{423} = 11g_1^4 g_2/6,\ b_{433} = 7g_1^5 g_2/4,\ b_{424} = 13g_1^6/36,\ b_{434} = g_1^7/4.$$

The coefficient of $N^{-i}$ or $n^{-i/a}$ in the expansion for the distribution of $m_n$, is a polynomial of degree $i$ in $\delta$ or $n^{1/a-1}$.

## Example

Let $F(x)$ be the gamma distribution with density $f(x) = x^{a-1}e^{-x}/\Gamma(a)$ on $[0, \infty)$ where $a > 0$. By 6.5.4, 6.5.29 of Abramowitz and Stegun (1964), $x^{-a}F(x)\Gamma(a) = \sum_{n=0}^{\infty}(-x)^n/(a+n)n!$. So (8) holds with $g_i = (-1)^n/\Gamma(a)(a+n)n!$ where $n = i - 1$:

$$g_1 = 1/a\Gamma(a) = 1/\Gamma(a+1),\ g_2 = -1/(a+1)\Gamma(a),\ g_3 = 1/2(a+2)\Gamma(a),$$
$$g_4 = -1/6(a+3)\Gamma(a),\ h_k = (-\Gamma(a+1))^k/k!,\ N = (n/\Gamma(a+1))^{1/a}.$$

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