Random graphs arising from Strong reinforcement models

Mark Holmes
(Joint work with C. Hirsch and V. Kleptsyn)

December 2017

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What kind of “neuronal architectures” can be obtained by simple reinforcement models?

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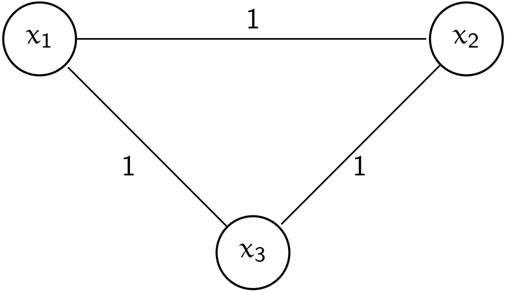
- Interested e.g. in the (random) subset $\mathcal{E}$ of edges used infinitely often/a positive proportion of the time.

Example: triangle



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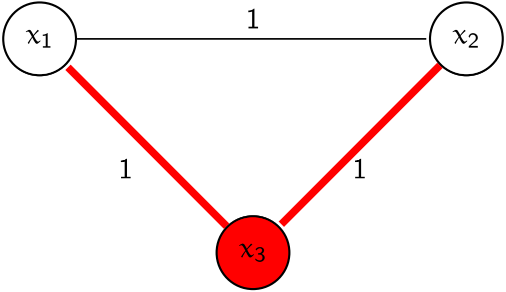

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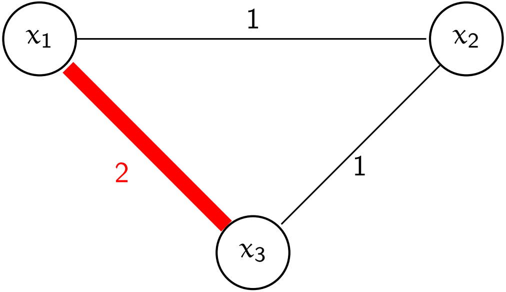

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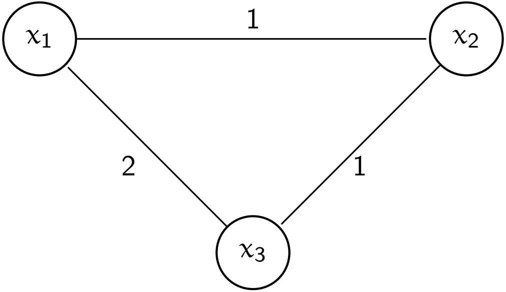

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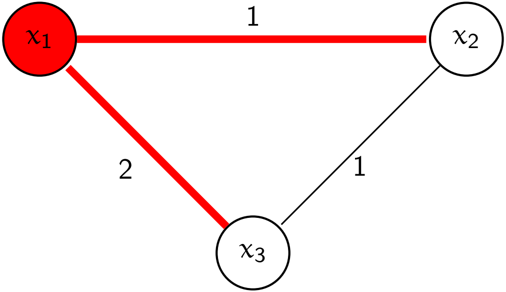

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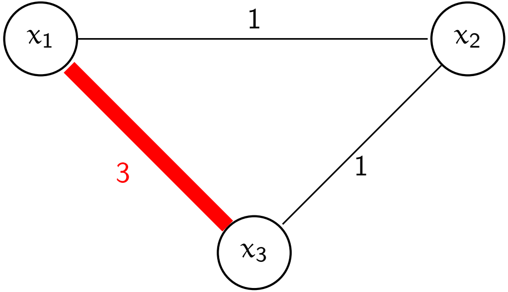

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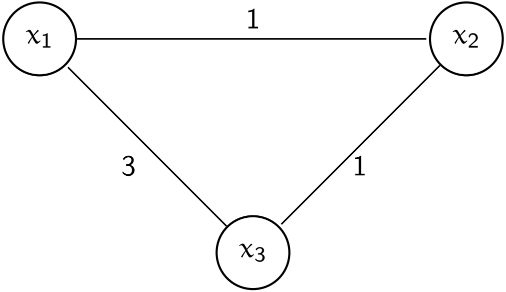

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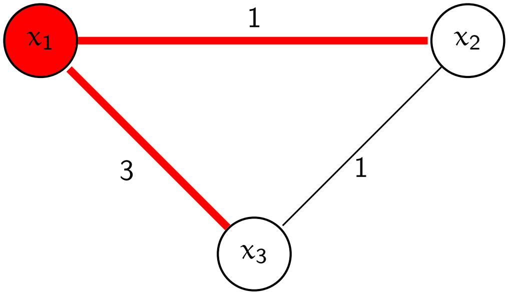

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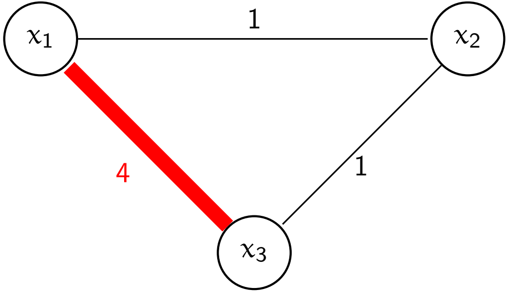

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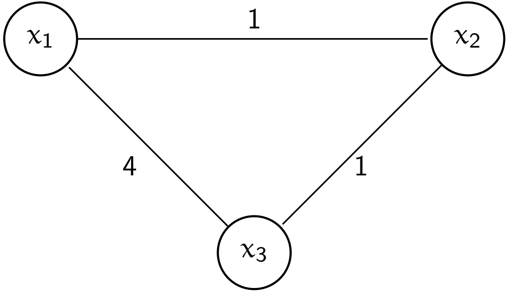

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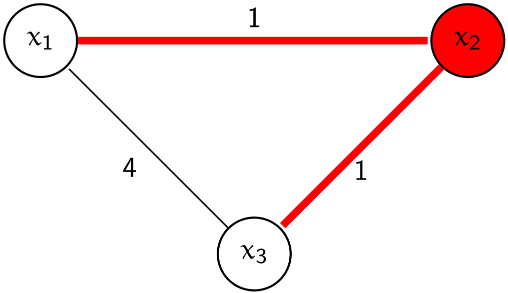
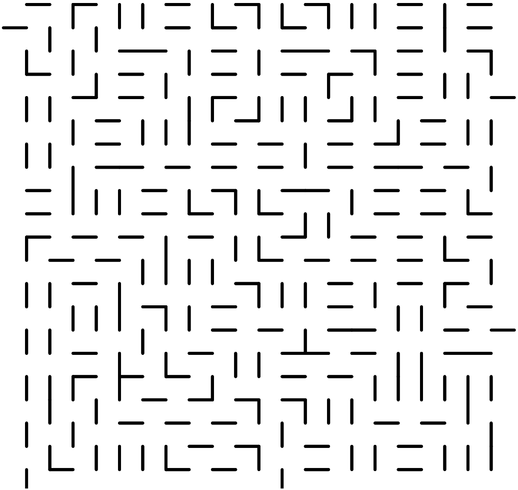


Example: $\mathbb{Z}^2$, $\alpha = 1.9$



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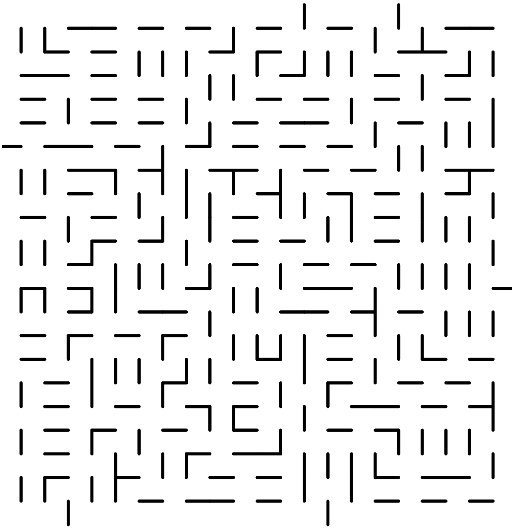

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v.d.Hofstad, H., Kuznetsov, Ruszel (2016), Basu and Benaim (??), H. Kleptsyn (2017), Hirsch, H. Kleptsyn (??)

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Theorem: (v.d.Hofstad, H., Kuznetsov, Ruszel) Any odd cycle becomes stable by taking α sufficiently close to 1.

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Moreover, any even length is possible by taking α sufficiently close to 1.

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- Process behaves independently on distinct components of $\mathcal{N}^c$.
- Process on a finite component F behaves *similarly to* WARM process on the graph F itself.

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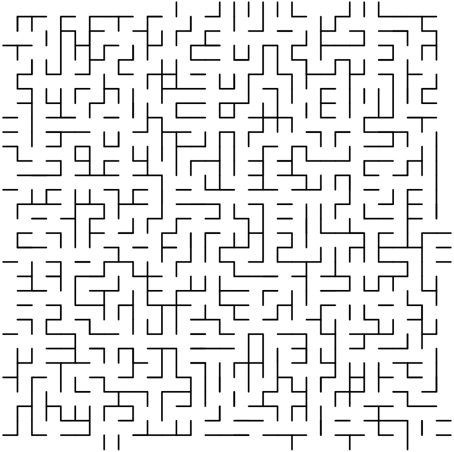
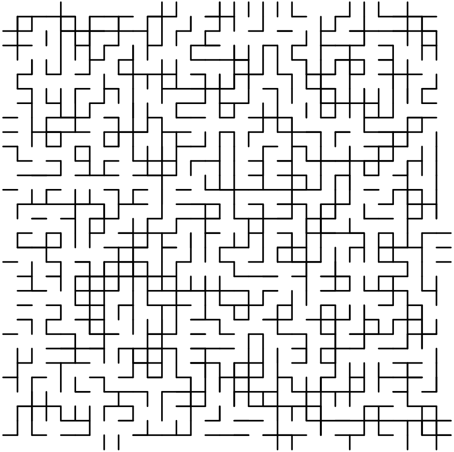


Bounded degree: Corrupt compass model

Now independently (at each vertex) with probability ε colour in all incident edges.



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- Once components of $\mathcal{N}^c$ are finite they are behaving like* finite components of a finite graph.

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- On the square lattice, no finite component of $\mathcal{E}$ contains an even cycle for any $\alpha > 1$.
- On the triangular lattice, you see cycles of odd length n if $\alpha < \frac{2}{1-\cos(\pi(1+(1/n)))}$.

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- Can force the walker to not get stuck on a single edge by never allowing it to backtrack.

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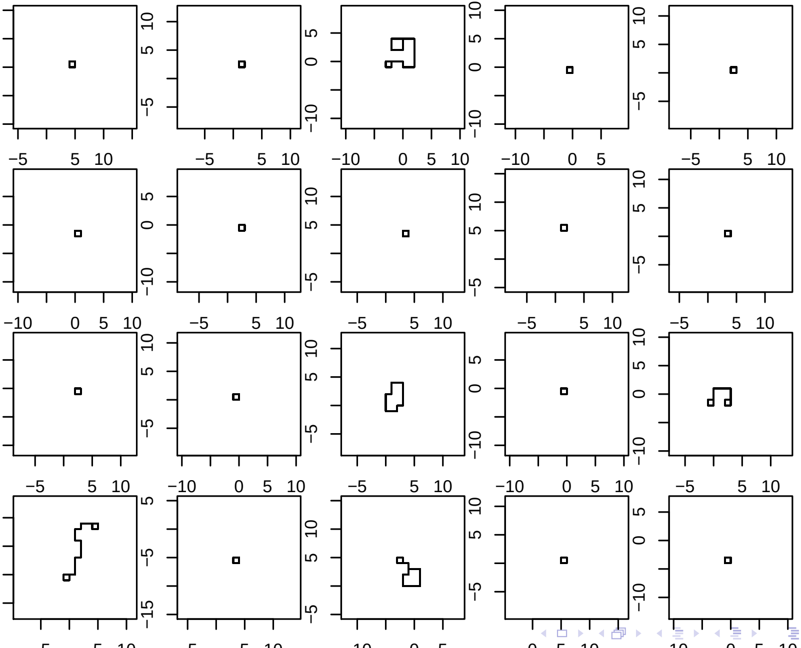
Q: Is there any finite graph (without leaves) on which the walker cannot get stuck for every $\alpha > 1$?

Simulations - $\alpha = 10$



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