

# Stability and uniform fluctuation estimates of Ensemble Kalman-Bucy filters

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6th Wellington Workshop Proba-Stats 2017

## Synthesis joint works:

- ▶ AAP-17 (Unif. EnKBF)+ Tugaut.
- ▶ SIAM Control & Opt.-17 (Unif. En-EKBF)+ Kurtzmann, Tugaut.
- ▶ SIAM Control & Opt.-17/18 (Stability KBF)+ Bishop
- ▶ SPA-17 (Perturbation KB)+ Bishop, S. Pathiraja.
- ▶ Arxiv 1 (Stability EKBF)+ Kurtzmann, Tugaut.
- ▶ Arxiv 2 (Stoch. Riccati)+ Bishop, A. Niclas.
- ▶ Arxiv 3 (1d-case)+ Bishop, Kamatani, Rémillard.

Kalman-Bucy filter

McKean-Vlasov interpretations

3 classes of algorithms

Mean field/Ensemble Kalman-Bucy filter

Stability of Kalman-Bucy diffusions

Stability of Riccati semigroup

Stability of stochastic flows

Uniform fluctuation estimates

Some observations/numerical issues

The Multivariate case

One dimensional filtering problems

Nonlinear models

Extended Kalman-Bucy-filters

Extended Ensemble Kalman-Bucy-filters

A stability theorem

Uniform propagation of chaos estimates

## Kalman-Bucy filter

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# Kalman-Bucy filter

## Linear+Gaussian filtering problem

$$\begin{cases} dX_t = A X_t dt + R^{1/2} dW_t \in \mathbb{R}^r \\ dY_t = C X_t dt + \Sigma^{1/2} dV_t \end{cases} \rightsquigarrow \mathcal{F}_t := \sigma(Y_s, s \leq t).$$

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## Optimal $\mathbb{L}_2$ -filter = Kalman-Bucy filter

$$\hat{X}_t := \mathbb{E}(X_t | \mathcal{F}_t) \quad \text{and} \quad P_t := \mathbb{E}((X_t - \mathbb{E}(X_t | \mathcal{F}_t))(X_t - \mathbb{E}(X_t | \mathcal{F}_t))')$$

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## State estimate

$$d\hat{X}_t = A \hat{X}_t dt + P_t C' \Sigma^{-1} (dY_t - C \hat{X}_t dt)$$

with the gain given by the Riccati equation

$$\partial_t P_t = \text{Ricc}(P_t) := A P_t + P_t A' - P_t \mathbf{S} P_t + R \quad \text{with} \quad \mathbf{S} := C' \Sigma C$$

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# Reformulation $\rightsquigarrow$ Nonlinear Kalman-Bucy diffusion

$\iff$  McKean-Vlasov type diffusions  $\bar{X}_t$  such that

$$\eta_t := \text{Law}(\bar{X}_t \mid \mathcal{F}_t) = \mathcal{N}[\hat{X}_t, P_t]$$

$\rightsquigarrow$  Interacting diffusions with their conditional covariance matrices

$$\mathcal{P}_{\eta_t} = \eta_t [(e - \eta_t(e))(e - \eta_t(e))'] \quad \text{with} \quad e(x) := x.$$

and such that

$$\mathbb{E}(X_t \mid \mathcal{F}_t) = \hat{X}_t \quad \text{and} \quad \mathcal{P}_{\eta_t} = P_t$$



## 3 classes of McKean-Vlasov type diffusions

### 1) "Vanilla EnKF" ( $\rightsquigarrow$ discrete time - Evensen 94)

$$d\bar{X}_t = A\bar{X}_t dt + R^{1/2} d\bar{W}_t + \mathcal{P}_{\eta_t} C' \Sigma^{-1} \left[ dY_t - \left( C\bar{X}_t dt + \Sigma^{1/2} d\bar{V}_t \right) \right]$$

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2) "deterministic EnKF" ( $\rightsquigarrow$  discrete time - Sakov-Oke 08)

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3) Pure transport equation (Reich-Cotter 13):

$$d\bar{X}_t = A \bar{X}_t dt + \frac{1}{2} (R \mathcal{P}_{\eta_t}^{-1} - \mathcal{P}_{\eta_t} S)(\bar{X}_t - \eta_t(e)) + \mathcal{P}_{\eta_t} C' \Sigma^{-1} [dY_t - C \eta_t(e) dt]$$

# The Ensemble Kalman-Bucy filter

**(Case 1) Mean field interpretation  $\rightsquigarrow N + 1$  interacting diffusions**

$$d\xi_t^i = A \xi_t^i dt + R^{1/2} d\bar{W}_t^i + p_t C' \Sigma^{-1} \left[ dY_t - \left( C \xi_t^i dt + \Sigma^{1/2} d\bar{V}_t^i \right) \right]$$

**with the rescaled particle covariance matrices**

$$p_t := \left( 1 + \frac{1}{N} \right) \mathcal{P}_{\eta_t^N} = \frac{1}{N} \sum_{1 \leq i \leq N+1} (\xi_t^i - m_t) (\xi_t^i - m_t)'$$

**and the empirical measures**

$$\eta_t^N := \frac{1}{N+1} \sum_{1 \leq i \leq N+1} \delta_{\xi_t^i} \quad \text{and the sample mean} \quad m_t := \frac{1}{N+1} \sum_{1 \leq i \leq N+1} \xi_t^i.$$

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*where is the Riccati equation?*

# Th1: The EnKF equations

## The EnKF equation

$$dm_t = A m_t dt + p_t C' \Sigma^{-1} (dY_t - C m_t dt) + \frac{1}{\sqrt{N+1}} d\bar{M}_t$$

with an  $r$ -dimensional martingale  $\bar{M}_t = (\bar{M}_t(k))_{1 \leq k \leq r}$  with angle-brackets

$$\partial_t \langle \bar{M} | \otimes | \bar{M} \rangle_t = U + p_t V p_t.$$

## With

- 1)  $(U, V) = (R, S)$     2)  $(U, V) = (R, 0)$     3)  $(U, V) = (0, 0)$

## The particle/ensemble Riccati equation

$$dp_t = \text{Ricc}(p_t) dt + \frac{1}{\sqrt{N}} dM_t$$

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Orthogonality property

$$\forall 1 \leq k, l, l' \leq r \quad \langle M(k, l), \overline{M}(l') \rangle_t = 0.$$

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## (Hyp.) Observability + Controllability

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$\Leftrightarrow$

$$d(\bar{X}_t - X_t)$$

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**Steady state:  $\exists! P > 0$  such that  $\text{Ric}(P) = 0$  and spectral abscissa**

$$\zeta(\mathbf{A} - \mathbf{P}\mathbf{S}) := \max \{ \text{Re}(\lambda) : \lambda \in \text{Spec}(\mathbf{A} - \mathbf{P}\mathbf{S}) \} < 0$$

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**STABLE EVEN WHEN  $A$  is unstable.**

# Bucy's theorems

## Notation (Peano-Baker-expo-type series) - State transition matrix

$$\mathcal{E}_t = \exp \left[ \int_0^t Q_u du \right] \implies \partial_t \mathcal{E}_t = Q_t \mathcal{E}_t$$



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**Theo [Bucy]**  $\exists v, \alpha, \beta > 0$  and  $\mathcal{G}_v^+ > \mathcal{G}_v^- > 0 : \forall P_0$  and  $t \geq v$

$$\mathcal{G}_v^- \leq P_t \leq \mathcal{G}_v^+ \quad \text{and} \quad \left\| \exp \int_0^t (A - P_u S) du \right\|_2^2 \leq \alpha \exp \{-\beta t\}$$

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*Also true for any  $t \geq 0$  with  $\alpha$  depending on  $P_0$ .*

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$$\begin{aligned} & \mathbb{E} (\|\psi_{s,t}(x_1, Q_1) - \psi_{s,t}(x_2, Q_2)\|_2^n \mid \mathcal{X}_s)^{1/n} \\ & \leq c_{Q_1, Q_2} e^{-\nu(t-s)} [\|x_1 - x_2\|_2 + \{\|x_2 - X_s\|_2 + \sqrt{n}\} \|Q_1 - Q_2\|_2] \end{aligned}$$

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**Observations:**

- ▶ same estimate for  $\bar{\psi}_{s,t}(x, Q) =$  stochastic flow Kalman-Bucy (nonlinear) diffusion
- ▶ similar estimate for  $\phi_t(Q) =$  semigroup of the Riccati equation

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Any choice

$$\nu \in \{ \beta, (1 - \epsilon) \varsigma(A - PS), \lambda_{\max}((A - PS)_{\text{sym}}) \}$$

is fine but  $c_{Q_1, Q_2}$  maybe larger than you expect.

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- ▶ **Case 3** = *Pure transport*  $\rightsquigarrow$  simple estimates using Lipschitz and contraction inequalities w.r.t. initial conditions

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$(m_t, X_t, \mathbf{p}_t) = (\text{sample mean, true signal, sample covariance})$

↓

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- ▶ **Time varying**  $\oplus$  **stochastic type** Ornstein-Uhlenbeck diffusion

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- ▶  $\mu(A)$  = Log-norm of  $A$ :

$\rightsquigarrow$  The matrix  $(A - pS)$  may be ill-conditioned in the sense that

$$\exists p : \mu(A - pS) = \lambda_{\max}((A - pS)_{\text{sym}}) > 0 > \lambda_{\min}((A - pS)_{\text{sym}})$$

## Multivariate case

**Hyp 1:**  $S > 0 \rightsquigarrow$  up a change of basis

$$(A, S) \rightsquigarrow (\bar{A}, \bar{S}) := (S^{1/2}AS^{-1/2}, I) \rightsquigarrow + \text{Hyp 2} \quad \mu(\bar{A}) < 0$$

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**Theo 1** [+Tugaut AAP-17] [Hyp 1 + Hyp 2]  $\forall n \geq 1 \exists N_n \geq 1 :$

$$N \geq N_n \implies \sup_{t \geq 0} \left[ \mathbb{E}(\|p_t - P_t\|^n) \vee \mathbb{E}(\|m_t - \hat{X}_t\|^n) \right] < c/\sqrt{N}$$

for the spectral of the Frobenius norm.



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**Theo 2** [+Bishop et al. 17] [only Observability + Controllability]

Uniform Riccati estimates even if we have the under-bias estimate

$$\forall t > 0 \quad 0 < p_t \quad \text{but} \quad 0 < \mathbb{E}(p_t) \leq P_t$$

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$$\forall t > 0 \quad 0 < p_t \quad \text{but} \quad 0 < \mathbb{E}(p_t) \leq P_t$$

+ Uniform CLT rates + Bias-Taylor type expansions + Robustness and Perturbations analysis (inflation, masking, shrinkage, projections, ...)

Kalman-Bucy filter

McKean-Vlasov interpretations

Stability of Kalman-Bucy diffusions

Uniform fluctuation estimates

**One dimensional filtering problems**

Nonlinear models

# 1d $\rightsquigarrow$ Closed form Riccati semigroups/tangent proc.

**Deterministic Riccati flow  $\phi_t(Q)$  on  $\mathbb{R}_+$ :**  $\text{Ricc}(\varpi_{\pm}) = 0$  for

$$S\varpi_- := A - \lambda/2 < 0 < S\varpi_+ := A + \lambda/2$$

with

$$\lambda = 2\sqrt{A^2 + RS}$$

$\Downarrow$

$$\sup_{Q \geq 0} \left[ |\phi_t(Q) - \varpi_+| \vee \exp\left(2 \int_0^t [A - \phi_s(Q)S] ds\right) \right] \leq c \exp(-\lambda t)$$

## Reversible measures

**Stochastic Riccati flow**  $Q_t := \Phi_t^\epsilon(Q_0) \in \mathbb{R}_+$  **with**  $\epsilon = 2/\sqrt{N}$ :

$$dQ_t \stackrel{\text{law}}{=} \text{Ricc}(Q_t)dt + \epsilon \sqrt{Q(U + QVQ)} dW_t$$

with  $\epsilon^2 U < 2R \implies$  origin repellent

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**Reversible measures**  $\pi_\epsilon(dx)$  on  $\mathbb{R}_+$ :

►  $U \wedge V > 0 \rightsquigarrow$  **Heavy tails**

$$\propto 1_{\mathbb{R}_+}(x) \frac{x^{\frac{2}{\epsilon^2} \frac{R}{U} - 1}}{[U + Vx^2]^{1 + \frac{1}{\epsilon^2} (\frac{R}{U} + \frac{S}{V})}} \exp \left[ \frac{4}{\epsilon^2} \frac{A}{\sqrt{UV}} \tan^{-1} \left( x \sqrt{\frac{V}{U}} \right) \right] dx.$$

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►  $U > V = 0 \rightsquigarrow$  **Gaussian-type tails**

$$\propto 1_{\mathbb{R}_+}(x) x^{\frac{2}{\epsilon^2} \frac{R}{U} - 1} \exp \left[ -\frac{S}{U\epsilon^2} \left( x - 2 \frac{A}{S} \right)^2 \right] dx.$$

## Stability Markov transition semigroup $P_t^\epsilon$

**Theo** [+Bishop, Kamatani, Rémillard 17]  $\forall A$  and  $\forall R \wedge S > 0$

- $\exists \zeta, \epsilon_0 > 0$  and some Wasserstein distance  $\mathbb{D}$  s.t. for any  $0 \leq \epsilon \leq \epsilon_0$

$$\mathbb{D}(\mu_1 P_t^\epsilon, \mu_2 P_t^\epsilon) \leq \exp(-\lambda (1 - \epsilon^2 \zeta) t) \mathbb{D}(\mu_1, \mu_2)$$



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## Some extensions

**Case 2:** If  $A < 0$  or  $0 < A < \sqrt{RS} \rightsquigarrow$  Poincaré inequalities (and  $\mathbb{L}_2(\pi_\epsilon)$ -contractions), ...

## Consequences

Uniform estimates for En-KF + particle Riccati diffusions, ...

Kalman-Bucy filter

McKean-Vlasov interpretations

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**Nonlinear models**

Extended Kalman-Bucy-filters

Extended Ensemble Kalman-Bucy-filters

A stability theorem

Uniform propagation of chaos estimates

# Nonlinear models

## Extended Kalman-Bucy-filters

$$d\hat{X}_t = A(\hat{X}_t) dt + P_t C' \Sigma^{-1} [dY_t - C\hat{X}_t dt]$$

with the "stochastic" Riccati equation:

$$\partial_t P_t = \partial A(\hat{X}_t) P_t + P_t \partial A(\hat{X}_t)' + R - P_t S P_t$$

# Nonlinear models

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## McKean-Vlasov interpretation

$$\begin{aligned} d\bar{X}_t = & \mathcal{A}(\bar{X}_t, \eta_t(e)) dt + R^{1/2} d\bar{W}_t \\ & + \mathcal{P}_{\eta_t} C' R_2^{-1} \left[ dY_t - \left( C\bar{X}_t dt + \Sigma^{1/2} d\bar{V}_t \right) \right] \end{aligned}$$

with the drift function

$$\mathcal{A}(x, m) := A[m] + \partial A[m] (x - m).$$

# Extended Ensemble Kalman-Bucy-filters

## En-EKF = Mean field particle model

$$d\xi_t^i = \mathcal{A}(\xi_t^i, m_t) dt + R^{1/2} d\bar{W}_t^i + p_t C' \Sigma^{-1} \left[ dY_t - \left( C \xi_t^i dt + \Sigma^{1/2} d\bar{V}_t^i \right) \right]$$

with the sample means  $m_t$  and covariance matrices  $p_t$  and the drift

$$\mathcal{A}(\xi_t^i, m_t) := A[m_t] + \underbrace{\frac{\partial A[m_t]}{\partial m_t} (\xi_t^i - m_t)}_{\text{Repulsion/Attraction w.r.t. } m_t}$$

# Some illustrations

## Langevin type signal processes

$$R = \sigma^2 Id \quad \text{and} \quad (A, \partial A) = (-\partial\mathcal{V}, -\partial^2\mathcal{V})$$

## Non quadratic potential ( $q \in \mathbb{R}^r, Q_1, Q_2 \geq 0$ )

$$\mathcal{V}(x) = \frac{1}{2} \langle Q_1 x, x \rangle + \langle q, x \rangle + \frac{1}{3} \langle Q_2 x, x \rangle^{3/2}$$

## Interacting diffusion gradient flows

$$\mathcal{V}(x) = \sum_{1 \leq i \leq r} \mathcal{U}_1(x_i) + \sum_{1 \leq i \neq j \leq r} \mathcal{U}_2(x_i, x_j)$$

for some convex confining potential  $\mathcal{U}_i : \mathbb{R}^i \mapsto [0, \infty[$

# Regularity conditions

**Full observation  $S = s Id$  and**

$$-\lambda_{\partial A} := \sup_{x \in \mathbb{R}^r} \lambda_{\max}(\partial A(x) + \partial A(x)') < 0$$

$$\|\partial A(x) - \partial A(y)\| \leq \kappa_{\partial A} \|x - y\|$$

**Examples: Langevin signal-diffusion**

$$(\lambda_{\partial A}, \kappa_{\partial A}) = \beta \left( 2^{-1} \lambda_{\min}(Q_1), 2 \lambda_{\max}^{3/2}(Q_2) \right).$$

*more generally  $\partial^2 \mathcal{V} \geq \nu Id \oplus$  Lipschitz condition*



# Stability theorem

$(\bar{X}_t, \bar{Z}_t) := \text{McKean-Vlasov starting at } (\bar{X}_0, \bar{Z}_0)$

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**Theo** [+Kurtzmann-Tugaut]

When  $\lambda_{\partial A}$  is sufficiently large we have

$$\mathbb{W}_2(\text{Law}(\bar{X}_t), \text{Law}(\bar{Z}_t)) \leq c \exp[-t \lambda] \quad \text{for some } \lambda > 0.$$

$\exists$  *more explicit description in terms of*  $(R, S, \kappa_{\partial A})$ .

# Propagation of chaos

$$\mathbb{P}_t^N := \text{Law}(m_t, p_t) \quad \mathbb{P}_t := \text{Law}(\widehat{X}_t, P_t)$$

and

$$\mathbb{Q}_t^N := \text{Law}(\xi_t^1) \quad \mathbb{Q}_t := \text{Law}(\overline{X}_t)$$

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↓

**Theo** [+Kurtzmann-Tugaut]

When  $\lambda_{\partial A}$  is sufficiently large,  $\exists \beta \in ]0, 1/2]$  s.t.

$$\sup_{t \geq 0} \mathbb{W}_2(\mathbb{P}_t^N, \mathbb{P}_t) \vee \sup_{t \geq 0} \mathbb{W}_2(\mathbb{Q}_t^N, \mathbb{Q}_t) \leq c N^{-\beta}$$

as soon as  $\text{tr}(P_0)$  is not too large and  $N$  large enough...